

LINEAR MAPS PRESERVING ORBITS

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ABSTRACT. Let $H \subset \mathrm{GL}(V)$ be a connected complex reductive group where V is a finite-dimensional complex vector space. Let $v \in V$ and let $G = \{g \in \mathrm{GL}(V) \mid gHv = Hv\}$. Following Raïs [Raï07] we say that the orbit Hv is *characteristic for H* if the identity component of G is H . If H is semisimple, we say that Hv is *semi-characteristic for H* if the identity component of G is an extension of H by a torus. We classify the H -orbits which are not (semi)-characteristic in many cases.

1. INTRODUCTION

Let K be a field. Then $H := \mathrm{PGL}_n(K)$ acts on $V := M(n, K)$ via conjugation. There is a large literature on solving *linear preserver problems*, that is, on finding the subgroups of $\mathrm{GL}(V)$ which preserve a certain set F of H -orbits in V . See [LP01] for a survey. One method of solving such problems is to classify all possible subgroups of $\mathrm{GL}(V)$ containing H and then check to see if these subgroups preserve F . This idea goes back at least to Dynkin [Dyn52] and has been used in many papers, e.g., [Gur94, Gur97, GL97, DP93, DL94, PD95]. We generalize the problem (but only in characteristic zero) by letting H be a reductive complex algebraic group, letting V be an arbitrary finite dimensional representation of H and letting F be an H -orbit Hv . The question then becomes: What is the subgroup G of $\mathrm{GL}(V)$ which preserves Hv ? The method of solution is often to look at the possible G and possible G_v such that $G = HG_v$ (which implies that $Gv = Hv$). We are able to answer the question in many circumstances. We are particularly interested in identifying those cases where G^0 is the image of H , which, in the language of Raïs [Raï07], means identifying those H -orbits which are *characteristic*.

Our base field is \mathbb{C} , the field of complex numbers. Let V be a finite dimensional H -module where H is a connected reductive group. Let $0 \neq v \in V$ and set $G := \{g \in \mathrm{GL}(V) \mid gHv = Hv\}$. Then G is a closed algebraic subgroup of $\mathrm{GL}(V)$ (see 2.1 below). We say that Hv is *characteristic for H* (or simply that v is characteristic for H or just that v is characteristic) if G^0 is the image of H in $\mathrm{GL}(V)$. (From now on we will not distinguish H from its image in $\mathrm{GL}(V)$, so we will say that v is characteristic if $G^0 = H$, even though this is not quite correct.) The definition that Hv is *semi-characteristic* is as above, except that we require only that G^0 is an extension of H by a torus (so G has to be reductive). In general, G is not reductive (see Examples 6.12, 6.13, 7.8 and 7.30). We say that v is *almost characteristic* if H is a Levi factor of G^0 and that v is *almost semi-characteristic* if H contains the semisimple part of a Levi factor of G^0 .

In §2 we consider some elementary properties of our definitions. We see that one has a chance for $G^0 = H$ only in the case that $v \in V$ is *generic*, which is equivalent to saying that Hv spans V . In §3 we consider what can happen to G if we add a trivial factor to V . We show that Hv is characteristic if H is a torus and $v \in V$ is generic. In §4 we consider the case that H is simple of rank at least 2 and V is irreducible. We recall some fundamental results of A. Onishchik which apply. We are then able to classify the irreducible H -modules V and $v \in V$ such that Hv is not semi-characteristic. We determine which orbits are semi-characteristic in the adjoint representation of a semisimple group. In §5 we consider the case that H is simple of rank at

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least 2 and V is reducible. We determine the possible semisimple G containing H such that $Gv = Hv$ for $v \in V$. In §6 we consider the case that H is semisimple and V is irreducible. In §7 we determine the structure of G when $H = \mathrm{SL}_2$. In an appendix we prove branching rules which we need to establish our results.

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2. ELEMENTARY REMARKS

We consider when we can remove the prefixes “almost” and “semi.” We also reduce to the case that Hv spans V . First we show that G is closed in $\mathrm{GL}(V)$.

Lemma 2.1. *Let V be a finite-dimensional H -module where H is algebraic. Let $G = \{g \in \mathrm{GL}(V) \mid gHv = Hv\}$. Then G is a closed subgroup of $\mathrm{GL}(V)$.*

Proof. Let $G_1 = \{g \in \mathrm{GL}(V) \mid g\overline{Hv} = \overline{Hv}\}$ and $G_2 = \{g \in \mathrm{GL}(V) \mid g(\overline{Hv} \setminus Hv) = (\overline{Hv} \setminus Hv)\}$. Then G_1 and G_2 are closed subgroups of $\mathrm{GL}(V)$ and $G = G_1 \cap G_2$. \square

We now consider complexifications of compact group actions. Let C be a compact Lie group and W a real C -module. Let $w \in W$ and assume that Cw spans W .

Proposition 2.2. *Let C , W and w be as above. Let $L = \{g \in \mathrm{GL}(W) \mid gCw = Cw\}$. Then L is compact.*

Proof. Fix a basis w_1, \dots, w_n of W lying in Cw and let $\|\cdot\|$ be a norm on W . Then for $g \in L$ and $1 \leq i \leq n$, $\|gw_i\|$ is bounded by a constant which is independent of g . Thus L is a closed bounded subset of $\mathrm{GL}(W)$, hence compact. \square

Corollary 2.3. *Let $H = C_{\mathbb{C}}$ be the complexification of C acting on $V = W \otimes_{\mathbb{R}} \mathbb{C}$. Let $G = \{g \in \mathrm{GL}(V) \mid gHw = Hw\}$. Then G is the complexification of L , hence reductive.*

Proof. Since Cw is real algebraic [Sch01, Lemma 4.3], it is defined by an ideal $I \subset \mathbb{R}[W]$, and clearly the complex zeroes of I are Hw . Let I_s denote the subspace of I of elements of degree at most s , $s \in \mathbb{N}$. Then I is generated by some I_s . Let f_1, \dots, f_m be a basis for I_s . Then $g \in \mathrm{GL}(W)$ lies in L if and only if $g^*f_i \in I_s$ for all i . This gives a set of real equations defining the compact Lie group L , and the complex solutions of these equations are $L_{\mathbb{C}}$. But the complex solutions of the equations are clearly G . Thus $G = L_{\mathbb{C}}$. \square

Recall ([Jac62b, Jac79, Ch. II Theorem 11]) that if $G \subset \mathrm{GL}(V)$ acts irreducibly on V , then G is reductive.

Corollary 2.4. *Let H be reductive, let V be an H -module and let $v \in V$. Suppose that v is almost semi-characteristic for H . Then v is semi-characteristic in the following two cases.*

- (1) V is an irreducible representation of H .
- (2) There is a compact Lie group C and real C -module W such that $V = W \otimes_{\mathbb{R}} \mathbb{C}$, $v \in W$ and $H = C_{\mathbb{C}}$.

The following result characterizes when Hv is a cone.

Proposition 2.5. *Let $0 \neq v \in V$ where V is an H -module. Suppose that Hv is a cone. Then there is a 1-parameter subgroup $\sigma: \mathbb{C}^* \rightarrow H$ such that v is an eigenvector of σ with nonzero weight.*

Proof. Since Hv is a cone, $v \in T_v(Hv)$ and there is an $X \in \mathfrak{h}$ such that $X(v) = v$. Applying an element of H we can assume that $X \in \mathfrak{b}$, a Borel subalgebra of \mathfrak{h} . Write $X = s + n$ (Jordan decomposition) where s is semisimple and n is nilpotent. Then s and n are in \mathfrak{b} . We can assume that $s \in \mathfrak{t} \subset \mathfrak{b}$ where \mathfrak{t} is the Lie algebra of T , a maximal torus of H . Write $v = \sum_{\lambda \in \Lambda} v_\lambda$ as a sum of nonzero weight vectors where Λ is the set of weights of V relative to T such that $v_\lambda \neq 0$. Let Φ be the set of positive roots. Then for $\lambda \in \Lambda$ we have $(s + n)v_\lambda = sv_\lambda$ modulo $\sum_{\mu > \lambda} V_\mu$ where $\mu > \lambda$ means that $\mu \in \lambda + \mathbb{N}\Phi$. Thus by an easy induction we get that $sv_\lambda = v_\lambda$ for all $\lambda \in \Lambda$ so that $sv = v$. Hence $S := \{t \in T \mid tv \in \mathbb{C}^*v\}^0$ is a subtorus of T which acts nontrivially on v . It follows that there is a one-parameter subgroup $\sigma: \mathbb{C}^* \rightarrow S$ as desired. \square

Proposition 2.6. *Let $0 \neq v \in V$ where V is an irreducible H -module. Suppose that $\mathbb{C}^*v \not\subset Hv$. Then v is characteristic if it is semi-characteristic. In particular, this holds if v is not in the null cone of V .*

Proof. The group G is reductive and its center is contained in the scalar matrices. Under our hypotheses on v , the center must be finite. \square

Let $V = \bigoplus_{i=1}^k n_i V_i$ be the isotypic decomposition of an H -module where H is nontrivial reductive. Let $v \in V$. Then $v = (v_{ij})$ where v_{ij} belongs to the j th copy of V_i , $j = 1, \dots, n_i$, $i = 1, \dots, k$. Let S denote $\mathrm{GL}(V)^H = \mathrm{GL}(n_1) \times \dots \times \mathrm{GL}(n_k)$. Let $U_i \subset V_i$ be the linear subspace of V_i generated by the v_{ij} , $j = 1, \dots, n_i$. If $\dim U_i = n_i$ for all i , then we say that v is *generic*.

Proposition 2.7. *Let H be reductive, let V be an H -module and let $v \in V$. Then the following are equivalent.*

- (1) *The span of Hv is V .*
- (2) *There is no nontrivial one parameter subgroup of $S = \mathrm{GL}(V)^H$ which fixes v .*
- (3) *The vector v is generic.*

Proof. If $s \in S$, then v satisfies one of the conditions if and only if sv does. Clearly, if (1) or (3) fails, we can find an s and an i such that $(sv)_{i1} = 0$. If W denotes the first copy of V_i in $n_i V_i$, we have that $\mathbb{C}^* = \mathrm{GL}(W)^H \subset \mathrm{GL}(V)^H$ is a one-parameter subgroup fixing v , so (2) fails. Conversely, if (2) fails, the fixed point set of a “bad” one-parameter subgroup is a proper H -submodule of V containing v and (1) and (3) fail. \square

Corollary 2.8. *Let $v \in V$ and let G be a Levi component of $\{g \in \mathrm{GL}(V) \mid gHv = Hv\}$. Then v is generic for H if and only if it is generic for G .*

If v is not generic, then G is in a rather trivial way larger than H . To avoid this case, we usually assume from now on that v is generic.

3. TRIVIAL FACTORS AND TORI

Let $v \in V$ and suppose that $V^H = (0)$. Consider $\tilde{v} = (v, 1) \in \tilde{V} := V \oplus \mathbb{C}$. Let $\tilde{G} = \{g \in \mathrm{GL}(\tilde{V}) \mid gH\tilde{v} = H\tilde{v}\}$. We conjecture that $\tilde{G} = G$, where $G \subset \mathrm{GL}(\tilde{V})$ in the canonical way. Equivalently, we conjecture that the subgroup of the affine group $\mathrm{Aff}(V)$ preserving Hv lies in $\mathrm{GL}(V)$. Note that v generic implies that \tilde{v} is generic (we can add at most a one-dimensional fixed point set). The following example shows that the conjecture fails if H is not reductive.

Example 3.1. Let $H = (\mathbb{C}, +)$ act on \mathbb{C}^2 by sending $(a, b) \in \mathbb{C}^2$ to $(a, ta + b)$, $t \in H$. Let $\tilde{H} = H \times \mathbb{C}$ where $(t, s) \cdot (a, b) = (a, ta + s + b)$, $(t, s) \in \tilde{H}$, $(a, b) \in \mathbb{C}^2$. Then for $a \neq 0$, the H and \tilde{H} orbits of (a, b) are the same, where $H \subset \mathrm{GL}(\mathbb{C}^2)$, $\tilde{H} \subset \mathrm{Aff}(\mathbb{C}^2)$ and $\tilde{H} \not\subset \mathrm{GL}(\mathbb{C}^2)$.

For this section only G will denote the subgroup of $\text{Aff}(V)$ preserving Hv (rather than the corresponding subgroup of $\text{GL}(V)$). It is easy to see that we can always reduce to the case that $V^H = (0)$, which we assume holds for the rest of this section.

We have a homomorphism $\text{Aff}(V) \rightarrow \text{GL}(V)$ which sends an element $(g, c) \in G \subset \text{GL}(V) \ltimes V$ to $g \in \text{GL}(V)$. Let G' denote the image of G in $\text{GL}(V)$.

Lemma 3.2. *The homomorphism $G \rightarrow G'$ is injective.*

Proof. The kernel K of $G \rightarrow G'$ consists of the pure translations in G , i.e., the homomorphisms $x \mapsto x + c$ where $x, c \in V$. Clearly K is isomorphic to a closed subgroup of the additive group $(V, +)$ of V . Now $(V, +)$ has Lie algebra V (trivial bracket) and the exponential map is the identity. Thus \mathfrak{k} is a vector subspace W of V and K/K^0 is isomorphic to a finite subgroup of $(V/W, +)$. Hence K is connected and $K = (W, +)$ where W must be H -stable. Let $\pi: V \rightarrow W$ be an H -equivariant projection (here we use that H is reductive). Then there are elements of G which translate v to v' where $\pi(v')$ is arbitrary. Since H preserves W and $\text{Ker } \pi$, this is not possible for elements of H , unless $W = 0$. Hence K is the trivial group. \square

Note that injectivity fails in the case of Example 3.1.

Lemma 3.3. *Let M be a reductive subgroup of the affine group $\text{Aff}(V)$. Then there is an $\alpha \in \text{Aff}(V)$ such that $\alpha M \alpha^{-1} \subset \text{GL}(V)$.*

Proof. We use transcendental methods. Choose a hermitian metric on V so that we have a unitary group $U(V) \subset \text{GL}(V)$. Let K be a maximal compact subgroup of M . Then M is the complexification $K_{\mathbb{C}}$ of K . Now any compact subgroup of $\text{Aff}(V)$ is contained in a maximal compact subgroup of $\text{Aff}(V)$ and all the maximal compact subgroups of $\text{Aff}(V)$ are conjugate [Hoc65, Ch. XV Theorem 3.1]. But clearly $U(V) \subset \text{Aff}(V)$ is maximally compact. Thus K is conjugate to a subgroup of $U(V)$, hence M is conjugate to a subgroup of $U(V)_{\mathbb{C}} = \text{GL}(V)$. \square

Proposition 3.4. *In the following cases $G \subset \text{GL}(V)$.*

- (1) *The image $G' \subset \text{GL}(V)$ is reductive.*
- (2) *There is an $h' \in H$ such that $h'v = \lambda v$, $\lambda \in \mathbb{C}$, $\lambda \neq 1$.*

Proof. If (1) holds, then G is reductive and there is an element $\alpha \in \text{Aff}(V)$ such that $\alpha G \alpha^{-1} \subset \text{GL}(V)$, hence $\alpha H \alpha^{-1} \subset \text{GL}(V)$. But one easily sees that any affine transformation conjugating H into $\text{GL}(V)$ must have translation part which is fixed by H . But V contains no nonzero H -fixed vectors. Hence $G \subset \text{GL}(V)$.

Assume (2). Let $x \mapsto c + A(x)$ be an element of $\mathfrak{g} = \text{Lie}(G)$ where $0 \neq c \in V$ and $A \in \mathfrak{gl}(V)$. Then the difference of $c + A(hv)$ and $c + A(hh'v)$ is a nonzero multiple of $A(hv)$ and lies in $\mathfrak{h}(hv)$ for any $h \in H$. Thus A itself lies in \mathfrak{g} and \mathfrak{g} contains the linear and translation parts of its elements. But \mathfrak{g} cannot contain pure translations, as we saw above. Thus $\mathfrak{g} \subset \mathfrak{gl}(V)$ and $G \subset \text{GL}(V)$. \square

Corollary 3.5. *We have that $G \subset \text{GL}(V)$ in the following cases.*

- (1) *V is an irreducible H -module.*
- (2) *V is an SL_2 -module whose irreducible components are all of even dimension, i.e., a module all of whose weights are odd.*

Remark 3.6. Suppose that $C, W, w \in W$ are as in Proposition 2.2 where $W^C = 0$. Let L denote the subgroup of the real affine group of W stabilizing Cw . Then one can show that L is compact, and as above one sees that $L \subset \text{GL}(W)$. Complexifying, we see that the subgroup of the affine group of $V = W \otimes_{\mathbb{R}} \mathbb{C}$ preserving Hw , where $H = C_{\mathbb{C}}$, is again just the complexification of L , a subgroup of $\text{GL}(V)$.

Proposition 3.7. *Let $V = \oplus_i n_i V_i$ be the isotypic decomposition of V . Suppose that for no i and j do we have that V_i occurs in $\text{Hom}(V_j, V_i)$. Then $G \subset \text{GL}(V)$.*

Proof. Suppose that $G \not\subset \text{GL}(V)$. Then we would have a subspace of \mathfrak{g} consisting of elements $A_w + w$, $w \in W$, where $W \simeq V_i$ is an irreducible submodule of V , $A_w \in \mathfrak{gl}(V)$ and $hA_w h^{-1} = A_{hw}$ for $h \in H$. Our hypotheses imply that A_w followed by projection to $n_i V_i$ is zero. Thus $\exp(A_w + w)(v)$ has the same projection to W as $v + w$. Hence we cannot have $Hv = Gv$. \square

Example 3.8. Let $V := \sum_{i=1}^n m_i \varphi_i$ and $H = A_n$, $n \geq 1$, where φ_i is the i th fundamental representation of H , $i = 1, \dots, n$. Then $G \subset \text{GL}(V)$.

Theorem 3.9. *Let H be a torus. Then*

- (1) $G \subset \text{GL}(V)$.
- (2) *If $v \in V$ is generic, then $G^0 = H$.*

Proof. We may assume that $V^H = (0)$. First consider (1) in the case that $H = \mathbb{C}^*$. Let W be the subspace of V spanned by $H \cdot v$. Then any $g \in G$ must preserve W , so we can replace V by W . Thus we can reduce to the case that $v \in V$ is generic. This implies that the weight spaces of H are one-dimensional. We have a weight basis v_1, \dots, v_n of V such that $v = (v_1, \dots, v_n)$ where the weight of v_i is $0 \neq a_i \in \mathbb{Z}$. Suppose that the orbit of v is preserved by a transformation (g, c) where $(g, c)(v) = (\sum_j a_{ij} v_j + c_i)$. Here the a_{ij} and c_i are scalars. Then the i th component of $g(\lambda \cdot v)$ (where λ is a parameter in $H = \mathbb{C}^*$) is $\sum_j a_{ij} \lambda^{a_j} v_j + c_i$. Now the powers of λ that occur are distinct, hence the Laurent polynomial in λ that gives the i th component has some nonzero coefficient for a nonzero power of λ . If $c_i \neq 0$, then one can see that the polynomial takes on the value 0 for some $\lambda \neq 0$. But the \mathbb{C}^* -orbit of v is nonzero in the i th slot. Thus $c_i = 0$ for all i and g lies in $\text{GL}(V)$ so we have (1). The reasoning above also shows that for each i there is a unique j such that $a_{ij} \neq 0$. Thus a power g^k of g preserves the weight spaces. Then $g^k v = hv$ for some $h \in H$, and it follows that $g^k = h$. Thus we have (2). Note that g normalizes $H = \mathbb{C}^*$, so that we actually have $g^2 \in H$.

Now suppose that H is a torus. As before, to prove (1), we can assume that v is generic. Let $(g = (a_{ij}), c) \in G$. Choose a 1-parameter subgroup λ of H such that all the characters of V , restricted to λ , are distinct. It follows, as above, that $c = 0$ and that a power of g lies in H . \square

3.10. Let G_0 denote a Levi component of G containing H . Then as we saw before, we must have that $G_0 \subset \text{GL}(V)$. We can write G' as $G_0 \ltimes G'_u$ where G'_u is the unipotent radical of G' . Then we have the corresponding decomposition of \mathfrak{g}' as $\mathfrak{g}_0 \ltimes \mathfrak{g}'_u$. As H -module, \mathfrak{g}'_u is completely reducible. Assuming that G is not contained in $\text{GL}(V)$ we can choose an irreducible H -module $W \subset \mathfrak{g}'_u$ whose inverse image in \mathfrak{g} is not contained in $\mathfrak{gl}(V)$. Then we have a copy of W in V and elements $A_w \in \mathfrak{gl}(V)$, $w \in W$, such that $x \mapsto A_w(x) + w$ lies in \mathfrak{g} and $\{A_w\}_{w \in W}$ maps to our copy of W in \mathfrak{g}'_u . For all $h \in H$ we have $hA_w h^{-1} = A_{hw}$.

Theorem 3.11. *Suppose that $v \in V$ is generic and in the null cone. Then $G \subset \text{GL}(V)$.*

Proof. Suppose the contrary. Let $V = \oplus_i n_i V_i$ be the isotypic decomposition of V as H -module. Let $A_w + w \in \mathfrak{g}$, $w \in W$ be as above where we may assume that $W = V_i$ (first copy) for some i . Let $\pi: V \rightarrow W$ be an equivariant projection and set $v_i = \pi(v)$. Since v is generic, $v_i \neq 0$. The projection of $\exp(w + A_w)(v)$ to W has the form $v_i + w + p(v, w)$ where $p(v, w)$ is a polynomial which has no linear factors in w and such that the coefficients of the various monomials in w are polynomials in v without constant term. By applying elements $h \in H$ we can make the coefficients of hw in $p(hv, hw)$ as small as we want. But there is no loss if we replace hw by w since we are able to consider all possible w . Thus we can assume that the coefficients of the monomials in w in $p(v, w)$ are very small, in which case the inverse function theorem tells us that $w \mapsto w + p(v, w)$ covers a ball around $0 \in W$ whose radius we can choose to be independent

of v (for v close to zero). Then we see that $w \mapsto v_i + w + p(v, w)$ takes on the value 0. Thus Gv contains a point which projects to $0 \in W$, which is impossible. Hence $G \subset \mathrm{GL}(V)$. \square

Recall that V is called *stable* if it contains a nonempty Zariski open subset of closed orbits.

Corollary 3.12. *Let V be stable with a one-dimensional quotient. Then $G \subset \mathrm{GL}(V)$.*

Proof. We have that $\mathbb{C}[V]^H = \mathbb{C}[f]$ where f is homogeneous of degree $d > 1$. Moreover, $f^{-1}(f(v)) = Hv = Gv$ if $f(v) \neq 0$. Now the case that $f(v) = 0$ follows from Theorem 3.11 and if $f(v) \neq 0$, then $Gv \supset \Gamma v$ where $\Gamma \subset G$ is a finite subgroup isomorphic to $\mathbb{Z}/d\mathbb{Z} \subset \mathbb{C}^*$ acting via scalar multiplication on V . Then $G \subset \mathrm{GL}(V)$ by Proposition 3.4(2). \square

Remark 3.13. A case by case check shows that H simple and $\dim V//H = 1$ implies that $G \subset \mathrm{GL}(V)$.

Theorem 3.14. *If $H = \mathrm{SL}_2$, then $G \subset \mathrm{GL}(V)$.*

We prove the theorem by contradiction, so assume that we have $A_w + w$ as in 3.10. Then the A_w lie in a Lie algebra of nilpotent matrices, and by Engel's theorem we can find a partial flag $0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$ such that V_1 is the joint kernel of the A_w , $V_2/V_1 \subset V/V_1$ is the joint kernel of the A_w , etc. Note that the V_j are H -stable.

Lemma 3.15. *We have $W \subset V_{k-1}$.*

Proof. Since $A_w(v) \in V_{k-1}$, we have that $(A_w + w)(v + V_{k-1}) = w + V_{k-1}$. Thus $\exp(A_w + w)(v + V_{k-1}) = v + w + V_{k-1}$. Let π be the projection of V to V/V_{k-1} with kernel V_{k-1} . If $W \not\subset V_{k-1}$, then $\pi(Gv)$ contains a nontrivial H -stable subspace of V/V_{k-1} . This is not possible for the H -orbit, hence $W \subset V_{k-1}$. \square

Lemma 3.16. *Suppose that for some $j \geq 1$ we have $W \subset V_{k-j}$ and $A_w(v) \in V_{k-j}$ for all $w \in W$. Then the stabilizer of $v + V_{k-j}$ in H is infinite.*

Proof. Suppose that $v + V_{k-j}$ has finite stabilizer. Since $A_w(v) + w$ projects to zero in V/V_{k-j} , we must have that $A_w(v) + w = 0$, else the G -orbit of v has dimension greater than $\dim H$. Now for $h \in H$, $A_w(hv) + w = h(A_{h^{-1}w} + h^{-1}w)(v) = 0$, so that the average of $A_w(hv) + w$ over a maximal compact subgroup K of H is zero. Since $V^H = 0$ and A_w is linear, the average of $w + A_w(hv)$ over K is w . Thus $W = 0$, a contradiction. \square

Lemma 3.17. *Suppose that $v + V_{k-j}$ is in the null cone of V/V_{k-j} for some $j \geq 1$. Then $W \subset V_{k-j-1}$.*

Proof. Assume that $W \not\subset V_{k-j-1}$. Our argument in 3.11 shows that the G -orbit of v projected to the image of W in V/V_{k-j-1} contains zero, which is not possible for the H -orbit. Hence $W \subset V_{k-j-1}$. \square

Lemma 3.18. *Suppose that for some $j \geq 1$ we have that $A_w(v) \in V_{k-j}$ for all $w \in W$ and that $W \subset V_{k-j}$. Then $v + V_{k-j}$ is in the null cone of V/V_{k-j} .*

Proof. Suppose not. Consider $V' := \mathbb{C} \cdot v + V_{k-j} \subset V$. Then for $z \in \mathbb{C}$, $z(A_w + w)$ exponentiates to an element $g(z)$ of $\mathrm{Aff}(V')$ which fixes $v' := v + V_{k-j} \in V/V_{k-j}$. By Lemma 3.16 we know that v' has an infinite stabilizer S in H . Since v' is not in the null cone, S has identity component $T \simeq \mathbb{C}^*$. For any $s \in S$ there is an $h_{z,s} \in H$ such that $h_{z,s}v = g(z)sv$. Then $h_{z,s} \in S$ since $g(z)s$ fixes v' so that the $g(z)$ preserve the S -orbit of v . The group generated by T and the $g(z)$ is connected, so it preserves the T -orbit of v . By Theorem 3.9 we see that $w = 0$, a contradiction. \square

Proof of Theorem 3.14. We have that $A_w(v) \in V_{k-1}$ and $W \subset V_{k-1}$. Suppose that we have $A_w(v) \in V_{k-j}$ and $W \subset V_{k-j}$ for some $j \geq 1$. By Lemmas 3.16 and 3.18 and genericity of v we may assume that $v + V_{k-j}$ is a sum of highest weight vectors. By Lemma 3.17, $W \subset V_{k-j-1}$, so that if $A_w(v) \in V_{k-j-1}$ we can continue. We eventually arrive at a case where $A_w(v) \notin V_{k-j-1}$ (we cannot have a pure translation in \mathfrak{g}). Since $v + V_{k-j}$ is a sum of highest weight vectors there are unique elements $B_w \in \mathfrak{u}$ such that $A_w(v) + w + B_w(v) \in V_{k-j-1}$. Here \mathfrak{u} is the Lie algebra of the standard unipotent subgroup of H . Since $A_w(v) + w \notin V_{k-j-1}$, $B_w(v) \notin V_{k-j-1}$ and $v + V_{k-j-1}$ is not a sum of highest weight vectors. Since $v + V_{k-j}$ is a nonzero sum of highest weight vectors, the H -isotropy group of $v + V_{k-j-1}$, which is a subgroup of the H -isotropy group of $v + V_{k-j}$, is finite. Arguing as in Lemma 3.16 we obtain that $w' := A_w(v) + w + B_w(v) = 0$ and that $W = 0$, a contradiction. Hence $G \subset \mathrm{GL}(V)$. \square

From now on we will assume that $V^H = 0$, even though we have only established our conjecture for SL_2 or the case that V is irreducible.

4. THE CASE H IS SIMPLE AND V IS IRREDUCIBLE

Our goal in this section is to find the possible $G \subset \mathrm{GL}(V)$ preserving an orbit Hv where V is an irreducible H -module and H is simple of rank at least two. We will see that perforce G is simple. We begin by recalling some important results of Onishchik.

Let $H \subset G$ where G and H are linear algebraic groups. Let V be an H -module. If $v \in V$ and $Gv = Hv$, then $G = HK$ where $K = G_v$. Conversely, $G = HK$ implies that $Gv = Hv$ for $v \in V^K$. There is a rather restricted class of possibilities for H and K when G is simple and H is semisimple, as follows from the work of Onishchik [Oni62, Oni69].

If K is a connected complex linear algebraic group, let \mathfrak{k} denote its Lie algebra and let $L(K)$ denote a Levi subgroup of K . The next two theorems follow from [Oni62] and [Oni69] (see also [Oni94] and [GO93]).

Theorem 4.1. *Let H and K be connected algebraic subgroups of the connected reductive group G . Then the following are equivalent.*

- (1) $G = HK$.
- (2) $G = \sigma(H)\tau(K)$ where σ and τ are any automorphisms of G .
- (3) $G = L(H)L(K)$.
- (4) $G_0 = H_0K_0$ where H_0 and K_0 are maximal compact subgroups of H and K contained in a maximal compact subgroup G_0 of G .
- (5) $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ (if H and K are reductive).

Corollary 4.2. *Suppose that $G = HK$ where all the groups are connected algebraic. Choose Levi factors $L(G) \supset L(H), L(K)$. Then $L(G) = L(H)L(K)$.*

Now assume that \mathfrak{h} and \mathfrak{k} are reductive subalgebras of the reductive Lie algebra \mathfrak{g} . Let \mathfrak{g}_s be the sum of the simple components of \mathfrak{g} of rank at least 2 (the *strongly semisimple* part of \mathfrak{g}). Let G_s be the corresponding subgroup of G . Let $r(\mathfrak{g})$ be the sum of the center and simple components of rank 1 of \mathfrak{g} so that $\mathfrak{g} = \mathfrak{g}_s \oplus r(\mathfrak{g})$.

Theorem 4.3. *Let \mathfrak{h} and \mathfrak{k} be reductive subalgebras of the reductive Lie algebra \mathfrak{g} . Then the following are equivalent.*

- (1) $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$.
- (2) $\mathfrak{g}_s = \mathfrak{h}_s + \mathfrak{k}_s$ and $r(\mathfrak{g})$ is the sum of the projections of $r(\mathfrak{h})$ and $r(\mathfrak{k})$ to $r(\mathfrak{g})$.

Corollary 4.4. *Suppose that $v \in V$ is not semi-characteristic and that H contains a strongly semisimple subgroup. Then so does G_v .*

From the above and [Oni62] we have the following

TABLE 1.

	G	H	$\varphi_1(G) H$	K	$\varphi_1(G) K$	$H \cap K$
1	A_{2n-1}	C_n	φ_1	A_{2n-2}	$\varphi_1 + \theta_1$	C_{n-1}
2	D_{n+1}	B_n	$\varphi_1 + \theta_1$	A_n	$\varphi_1 + \varphi_n$	A_{n-1}
3.1	D_{2n}	B_{2n-1}	$\varphi_1 + \theta_1$	C_n	$2\varphi_1$	C_{n-1}
3.2	D_{2n}	B_{2n-1}	$\varphi_1 + \theta_1$	$C_n \times A_1$	$\varphi_1 \otimes \varphi_1$	$C_{n-1} \times A_1$
4.1	B_3	G_2	φ_2	B_2	$\varphi_1 + \theta_2$	A_1
4.2	B_3	G_2	φ_2	D_3	$\varphi_1 + \theta_1$	A_2
5.1	D_4	B_3	φ_3	B_2	$\varphi_1 + \theta_3$	A_1
5.2	D_4	B_3	φ_3	$B_2 \times A_1$	$\varphi_1 + \varphi_1^2$	$A_1 \times A_1$
5.3	D_4	B_3	φ_3	D_3	$\varphi_1 + \theta_2$	A_2
5.4	D_4	B_3	φ_3	B_3	$\varphi_1 + \theta_1$	G_2
6	D_8	B_7	$\varphi_1 + \theta_1$	B_4	φ_4	B_3

Theorem 4.5. *Let G be connected, simple and simply connected of rank at least 2. Let H and K be connected semisimple subgroups of G such that $G = HK$. Then, up to switching the roles of H and K and replacing each of them by their image under an automorphism of G , all possibilities are listed in Table 1.*

In our tables, we always have $n > 1$ and $k \geq 1$. We use θ_k to denote a trivial representation of dimension k . Corresponding to an ordering of the simple roots of G we have fundamental representations $\varphi_i = \varphi_i(G)$, $i = 1, \dots, \text{rank } G$. We use the ordering of the roots of the simple groups of Dynkin [Dyn52]. Note that entries (5.1), (5.2) and (5.3) of Table 1 are special cases of (3.1), (3.2) and (2), but we have included them for completeness.

Corollary 4.6. *Let (G, H, K) be a triple in Table 1.*

- (1) *If $L \subset G$ is a reductive subgroup commuting with H or K , then L has rank at most 1.*
- (2) *We have $G = H_s K_s$ where K_s and H_s are simple.*

Now that we know the possibilities for G , H and K , our task is to find the irreducible representations of G which remain irreducible when restricted to H . This can be read off from [Dyn52, Table 5]. However, given that one knows the possibilities for (G, H, K) , it is relatively easy to see which irreducible representations of G are possible. Note that we can sometimes gain an irreducible representation by adding a group of rank 1 to H (Table 2(3.5)).

Theorem 4.7. *Let $G = G_s$ be simple and let H and $K = K_s$ be proper semisimple subgroups of G such that $G = HK$. Assume that V is an irreducible representation of G which is also irreducible when restricted to H . Then, up to automorphisms of G , all possibilities are listed in Table 2.*

In Table 2, if V is a K -module, then $\mathcal{S}^k(V)$ denotes the K -subspace of $S^k(V)$ generated by $S^k(V^U)$ where U is a maximal unipotent subgroup of K . In other words, in $\mathcal{S}^k(V)$ we take only the Cartan components of products. In column V^K the notation $S(f_2, f_3)_k$ means the span of the monomials in f_2 and f_3 of degree k where f_i has degree i , $i = 2, 3$. Here the $f_i \in \mathbb{C}[\varphi_8(D_8)]^{B_4}$. A similar interpretation applies to $S(f_4)_k$ where $f_4 \in \mathbb{C}[\varphi_7(D_8)]^{B_4}$ has degree 4. The justification of the entries $V|K$ and V^K can be found in the Appendix.

Remark 4.8. In Table 2, we have chosen not to remove all redundancies due to automorphisms of groups of type D_n . In column $V|K$ we have omitted the decompositions in (3.2) and (3.4) which are obtained by restricting V to C_n . To determine this one needs the branching rule for restrictions of SL_{2n} -representations to C_n . As determined by Weyl [Wey39], one proceeds as

TABLE 2.

	G	V	H	$V H$	K	$V K$	V^K
1	A_{2n-1}	φ_1^k	C_n	φ_1^k	A_{2n-2}	$\varphi_1^k + \varphi_1^{k-1} + \cdots + \theta_1$	θ_1
2.1	D_{2n+1}	φ_{2n}^k	B_{2n}	φ_{2n}^k	A_{2n}	$S^k(\varphi_1 + \varphi_3 + \cdots + \varphi_{2n-1} + \theta_1)$	θ_1
2.2	D_{2n+1}	φ_{2n+1}^k	B_{2n}	φ_{2n}^k	A_{2n}	$S^k(\varphi_2 + \varphi_4 + \cdots + \varphi_{2n} + \theta_1)$	θ_1
3.1	D_{2n}	φ_{2n-1}^k	B_{2n-1}	φ_{2n-1}^k	A_{2n-1}	$S^k(\varphi_2 + \varphi_4 + \cdots + \varphi_{2n-2} + \theta_2)$	$S^k(\mathbb{C}^2)$
3.2	D_{2n}	φ_{2n-1}^k	B_{2n-1}	φ_{2n-1}^k	C_n	*	$S^k(\mathbb{C}^{n+1})$
3.3	D_{2n}	φ_{2n}^k	B_{2n-1}	φ_{2n-1}^k	A_{2n-1}	$S^k(\varphi_1 + \varphi_3 + \cdots + \varphi_{2n-1})$	(0)
3.4	D_{2n}	φ_{2n}^k	B_{2n-1}	φ_{2n-1}^k	C_n	*	(0)
3.5	D_{2n}	φ_1	$C_n \times A_1$	$\varphi_1 \otimes \varphi_1$	B_{2n-1}	$\varphi_1 + \theta_1$	θ_1
4.1	B_3	φ_1^k	G_2	φ_2^k	B_2	$S^k(\varphi_1 + \theta_2)$	$S^k(\mathbb{C}^2)$
4.2	B_3	φ_1^k	G_2	φ_2^k	D_3	$S^k(\varphi_1 + \theta_1)$	θ_1
5.1	D_4	φ_1^k	B_3	φ_3^k	B_2	$S^k(\varphi_1 + \theta_3)$	$S^k(\mathbb{C}^3)$
5.2	D_4	φ_1^k	B_3	φ_3^k	D_3	$S^k(\varphi_1 + \theta_2)$	$S^k(\mathbb{C}^2)$
5.3	D_4	φ_1^k	B_3	φ_3^k	B_3	$S^k(\varphi_1 + \theta_1)$	θ_1
5.4	D_4	φ_1	$C_2 \times A_1$	$\varphi_1 \otimes \varphi_1$	B_3	$\varphi_1 + \theta_1$	θ_1
6.1	D_8	φ_1	B_4	φ_4	B_7	$\varphi_1 + \theta_1$	θ_1
6.2	D_8	φ_7	B_4	$\varphi_1 \varphi_4$	B_7	φ_7	(0)
6.3	D_8	φ_7^k	B_7	φ_7^k	B_4	*	$S(f_4)_k$
6.4	D_8	φ_8^k	B_7	φ_7^k	B_4	*	$S(f_2, f_3)_k$

follows. Let $\omega \in \bigwedge^2((\mathbb{C}^{2n})^*)$ be nonzero and C_n -invariant. Then from the exterior powers of ω we obtain invariants in the duals of $\varphi_2^{b_2} \cdots \varphi_{2n-2}^{b_{2n-2}}$ for nonnegative b_j . Then the restriction of an irreducible representation φ of SL_{2n} to C_n is obtained by taking all possible complete contractions of the duals of our invariants in the $\varphi_2^{b_2} \cdots \varphi_{2n-2}^{b_{2n-2}}$ with φ . For example, $\varphi_1 \varphi_5(SL_8)$ contracted with ω gives rise to φ_4 and $\varphi_1 \varphi_3$ while contraction with $\omega \wedge \omega \in \bigwedge^4(\mathbb{C}^8)^*$ gives rise to φ_1^2 and φ_2 . Note that the only representations of SL_{2n} which can give rise to the trivial representation of C_n are those of the form $\varphi_2^{a_2} \cdots \varphi_{2n-2}^{a_{2n-2}}$. Hence in Table 2 (3.4) the trivial C_n -representation does not occur in the column $V|K$ for any $k \geq 1$ and in (3.2) the occurrences of the trivial representation are the symmetric algebra in θ_2 and the subspaces $\varphi_2^{C_n}, \dots, \varphi_{2n-2}^{C_n}$.

Remark 4.9. We do not know what to put in the column $V|K$ in cases (6.3) and (6.4) of Table 2. However, in the Appendix we are able to compute V^K .

Suppose that $H \subset G$ where G and H are semisimple, and connected and V is a G -module which is irreducible as an H -module. Then the inclusion of H in G has a very special form, as shown by Dynkin [Dyn52, Theorem 2.2].

Theorem 4.10. *Let G_1, \dots, G_k be the simple components of G . Then $H = H_1 \cdots H_k$ where the H_i are nontrivial semisimple subgroups of the G_i , $i = 1, \dots, k$.*

We are interested in the case that H is simple. Then Theorem 4.10 tells us that G has to be simple and from Table 2 we get the following theorem.

Theorem 4.11. *Let H be simple of rank at least two and let $\varphi: H \rightarrow GL(V)$ be an irreducible representation. Then every nonzero H -orbit Hv is semi-characteristic, except for the following cases (where $n \geq 2$ and $k \geq 1$).*

- (1) $H = C_n$, $\varphi = \varphi_1^k$ and v is a highest weight vector. Equivalently, v is fixed by A_{2n-2} where A_{2n-2} , C_n , A_{2n-1} and V are as in Tables 1(1) and 2(1).
- (2) $H = B_{2n}$, $\varphi = \varphi_{2n}^k$ and v is a highest weight vector. Equivalently, v is fixed by A_{2n} where A_{2n} , B_{2n} , D_{2n+1} and V are as in Tables 1(2) and 2(2.1) or 2(2.2).

- (3) $H = \mathbf{B}_{2n-1}$, $\varphi = \varphi_{2n-1}^k$ and v is fixed by \mathbf{C}_n where \mathbf{C}_n , \mathbf{B}_{2n-1} , \mathbf{D}_{2n} and V are as in Tables 1(3.1) and 2(3.2).
- (4) $H = \mathbf{G}_2$, $\varphi = \varphi_2^k$ and v is fixed by \mathbf{B}_2 where \mathbf{B}_2 , \mathbf{G}_2 , \mathbf{B}_3 and V are as in Tables 1(4.1) and 2(4.1).
- (5) $H = \mathbf{B}_4$, $\varphi = \varphi_4$ and Hv is closed. Equivalently, v is fixed by \mathbf{B}_7 where \mathbf{B}_4 , \mathbf{B}_7 , \mathbf{D}_8 and V are as in Tables 1(6) and 2(6.1).
- (6) $H = \mathbf{B}_7$, $\varphi = \varphi_7^k$ and v is fixed by \mathbf{B}_4 where \mathbf{B}_4 , \mathbf{B}_7 , \mathbf{D}_8 and V are as in Tables 1(6) and 2(6.3) or 2(6.4).

For special direct sums of representations we have the following result.

Proposition 4.12. *Let V_i be an irreducible H_i -module where the H_i are semisimple, $i = 1, \dots, k$. Let $V = V_1 \oplus \dots \oplus V_k$ be the corresponding $H = H_1 \times \dots \times H_k$ -module. Suppose that $0 \neq v_i \in V_i$ such that v_i is (semi)-characteristic for H_i for each i . Then v is (semi)-characteristic for H where $v = (v_1, \dots, v_k) \in V$.*

Proof. Let G be as usual. First suppose that V is an irreducible G^0 -module. Then, up to a cover and scalar matrices, $G^0 = G_1 \times \dots \times G_r$ where the G_j are simple and $V \simeq U_1 \otimes \dots \otimes U_r$ where the U_j are irreducible G_j -modules. By Theorem 4.10 each simple factor of each H_i must project nontrivially to a single G_j . But given the structure of V as H -module, this implies that $k = 1$, where the theorem is trivial.

We may now assume that there is a maximal flag $W_1 \subset \dots \subset W_r \subset W_{r+1} = V$ of G^0 -stable subspaces where $r \geq 1$. We may assume that, as H -module, $W_r = V_1 \oplus \dots \oplus V_p$ so that $V/W_r \simeq V_{p+1} \oplus \dots \oplus V_k$. The image of G in $\mathrm{GL}(V/W_r)$ is reductive. Let G' denote its semisimple part. Then for $g \in G'$, we have $g(v_{p+1}, \dots, v_k) \in (H_{p+1} \times \dots \times H_k)(v_{p+1}, \dots, v_k)$. By induction on k , $G' = H_{p+1} \times \dots \times H_k$. But by maximality of the flag, we must have that $p = k - 1$, i.e., $V/W_r \simeq V_k$. If V_k is not G -stable, then \mathfrak{g} contains a nonzero linear map of V_k to W_r . Since \mathfrak{g} is stable under the action of H , we may assume that it contains $\mathrm{Hom}(V_k, V_1)$. Thus the G -orbit of v contains a point $(0, v_2, \dots, v_k)$. Such a point is not in Hv , so we have a contradiction. Thus V_k is G -stable and we have a G -module direct sum decomposition $V = W_r \oplus V_k$. It follows by induction on k that Hv is (semi)-characteristic. \square

If one considers the adjoint representation \mathfrak{h} of a simple H , the only case that appears in Theorem 4.11 is $H = \mathbf{C}_n$, $n \geq 2$, where $\mathfrak{h} = \varphi_1^2$. Thus we have

Corollary 4.13. *Let $H = H_1 \times \dots \times H_k$ where the H_i are simple, and let $\varphi: H \rightarrow \mathrm{GL}(\mathfrak{h})$ be the adjoint representation. Let $v = (v_1, \dots, v_k) \in \oplus_i \mathfrak{h}_i$ where no v_i is zero. Then v is semi-characteristic if and only if for every simple component H_i of type \mathbf{C}_n , $n \geq 2$, $v_i \in \mathfrak{c}_n$ is not on the highest weight orbit.*

5. THE CASE H IS SIMPLE

We now consider the case where H is simple of rank at least two and our H -module V may be reducible. We consider the possible semisimple G which can act on V such that $Gv = Hv$ where $v \in V$ is generic for the action of H .

Here are some examples to keep in mind.

Example 5.1. Let H be simple and let V be an H -module. Let $G = H \times H$ and let v be the identity in $V \otimes V^*$. Then $Gv = Hv$ where the diagonal copy of H in G plays the role of K .

Example 5.2. Let (G, H, K) or (G, K, H) be an entry of Table 1. Let $V = \bigoplus_{i=1}^n m_i V_i$ be the isotypic decomposition of a G -module such that $\dim V_i^K \geq m_i \geq 1$ for all i . Let $v \in V^K$ be generic for the action of G . Then $Gv = Hv$.

Example 5.3. Here $H \simeq A_{2n-1}$. Let $(G_1, H_1, K_1) = (D_{2n}, A_{2n-1}, B_{2n-1})$ and $(G_2, H_2, K_2) = (A_{2n-1}, A_{2n-2}, C_n)$. Let $G = G_1 \times G_2$, let H denote the diagonal copy of A_{2n-1} and let $K = K_1 \times K_2$. Then $G = HK$. Let V be a representation of G which contains a generic vector $v \in V^K$. If V is irreducible, then $V = V_1 \otimes V_2$ where V_i is an irreducible representation of G_i , $i = 1, 2$, and $v \in V_1^{K_1} \otimes V_2^{K_2}$. The only possibilities allowing nontrivial fixed points are $V_1 = \varphi_1^k$, $k \geq 0$, and $V_2 = \varphi_2^{a_2} \varphi_4^{a_4} \cdots \varphi_{2n-2}^{a_{2n-2}}$ where the a_{2i} are in \mathbb{Z}^+ . In both cases, $\dim V_i^{K_i} = 1$. Thus for v to be generic, V must be a sum of representations (each of multiplicity one) of the form $V_1 \otimes V_2$ where $\dim V_i^{K_i} = 1$ for all i . For V to be almost faithful the sum must contain a nontrivial V_1 and a nontrivial V_2 .

Example 5.4. Here $H \simeq A_3$. Let $(G_1, H_1, K_1) = (B_3, D_3, G_2)$ and $(G_2, H_2, K_2) = (A_3, A_2, C_2)$. Let $G = G_1 \times G_2$, let H be the diagonal copy of $A_3 = D_3$ and let $K = K_1 \times K_2$. Then $G = HK$. If V_i is an irreducible representation of G_i , $i = 1, 2$, with $V_1^{K_1} \neq (0) \neq V_2^{K_2}$, then $V_1 = \varphi_3^k$, $k \geq 0$ and $V_2 = \varphi_2^\ell$, $\ell \geq 0$. Again, $\dim V_i^{K_i} = 1$, $i = 1, 2$, and the conditions for v generic and V almost faithful are as in the case above.

Example 5.5. Here $H \simeq B_3$. Let $(G_1, H_1, K_1) = (D_4, B_3, B_3)$ and $(G_2, H_2, K_2) = (B_3, G_2, B_2 \text{ or } D_3)$. Let $G = G_1 \times G_2$, let $H \subset H_1 \times G_2$ be the diagonal copy of B_3 and let $K = K_1 \times K_2$. Then $G = HK$. The possibilities for the V_i having nontrivial K_i -fixed points are $V_1 = \varphi_1^k$, $k \geq 0$ and $V_2 = \varphi_1^a \varphi_2^b$ if $K_2 = B_2$ and $V_2 = \varphi_1^a$ if $K_2 = D_3$ where a and b are nonnegative. While $V_1^{K_1}$ has dimension 1, this is not true for $V_2^{K_2}$, in general, if $K_2 = B_2$. Let $v \in V$ be generic and K -fixed. Then irreducible G -modules which can occur in V are sums of tensor products of modules $V_1 \otimes V_2$ where $\dim V_i^{K_i} \geq 1$, $i = 1, 2$.

Example 5.6. Let (G_1, H_1, K_1) be an entry of Table 1. Let $G = G_1 \times G_1$, let H be the diagonal copy of G_1 and let $K = H_1 \times K_1 \subset G$. Then $G = HK$. It is usually easy to determine the almost faithful $V_1 \otimes V_2$ with K -fixed points. For example, in the case of $(A_{2n-1}, C_n, A_{2n-2})$, V_1 has to be of the form $\varphi_2^{a_2} \cdots \varphi_{2n-2}^{a_{2n-2}}$ where the a_{2i} are nonnegative and V_2 has to be of the form φ_1^k or φ_{2n-1}^k for $k \geq 0$. On the other hand, if we have (D_8, B_7, B_4) , then V_1 is of the form φ_1^k , $k \geq 0$, and we have been unable to pin down exactly which V_2 have B_4 fixed points.

Looking at Table 1 one easily sees the following.

Proposition 5.7. *Suppose that (G, H, K) and (G', H', K') appear in Table 1 where $H \cap K \simeq H' \subset L \subset G'$ or $H \cap K \simeq K' \subset L \subset G'$ and L is isomorphic to H or K . Then G' is isomorphic to L .*

5.8. Let H be simple of rank at least 2 and let V be an almost faithful H -module. Let $v \in V$ be generic. Let $G = G_1 \times \cdots \times G_r$ where the G_i are simple and simply connected and G acts almost faithfully on V such that $Gv = Hv$ where $H \subset G$. Let K denote a Levi factor of G_v . Then $G = HK$. Let $\text{pr}_i: G \rightarrow G_i$ denote projection on the i th factor, $i = 1, \dots, r$. We may assume that $\text{pr}_i(H) \neq \{e\}$ for $1 \leq i \leq s$ and $\text{pr}_i(H) = \{e\}$ for $s < i \leq r$ where $s \geq 1$. Let $G' = G_1 \times \cdots \times G_s$ and $G'' = G_{s+1} \times \cdots \times G_r$. For $r \geq j > s$ there is a unique simple component K_j of K such that $\text{pr}_j(K_j) = G_j$ and clearly $K'' := K_{s+1} \times \cdots \times K_r$ covers G'' . The kernel of $K'' \rightarrow G'$ commutes with H and fixes v . Since Hv spans v (Proposition 2.7), the kernel must be finite. Hence K'' covers its image in G' . Let K' denote the product of the simple components of K not in K'' . Then $K' \subset G'$ and the projection of K'' to G' centralizes K' . We must have that $HK' = G'$.

We may write $V = \bigoplus V_i \otimes W_i$ where the V_i are pairwise nonisomorphic irreducible representations of G' and the W_i are representations of G'' . Then the projection v_i of v to each $V_i \otimes W_i$ is a tensor of rank $\dim W_i$ since v is generic. Let U_i denote the smallest subspace of V_i such that $v_i \in U_i \otimes W_i$. Then $\dim U_i = \dim W_i$ and $v_i \in U_i \otimes W_i$ corresponds to a K'' -equivariant

isomorphism of W_i^* onto $U_i \subset V_i$. In the sense of the following definition, v_i corresponds to a *subordination* $\alpha_i: (W_i^*, G'') \rightarrow (V_i, G')$.

Definition 5.9. Let Z_i be an L_i -module $i = 1, 2$, where the L_i are reductive. We say that Z_1 is *subordinate* to Z_2 if there is a linear injection $\alpha: Z_1 \rightarrow Z_2$ and a reductive subgroup $L \subset L_1 \times L_2$ such that α is L -equivariant (for the L -module structures on Z_1 and Z_2). Moreover, we require that $\text{pr}_1: L \rightarrow L_1$ be a cover. We say that $\alpha: Z_1 \rightarrow Z_2$ is a *subordination* of Z_1 to Z_2 . We sometimes use the notation $\alpha: (Z_1, L_1) \rightarrow (Z_2, L_2)$ to specify the groups involved.

We now consider the possibilities for K' .

Lemma 5.10. *Let H , etc. be as in (5.8). Then for $1 \leq i \leq s$ we have $\text{pr}_i(K') \neq G_i$.*

Proof. Suppose that $\text{pr}_i(K') = G_i$. Then there is a unique simple component K_i of K' such that $\text{pr}_i(K_i) = G_i$. If $\text{pr}_j(K_i) = \{e\}$ for $j \neq i$, then G_i acts trivially on Gv , which is not possible. Hence $\text{pr}_j(K_i) \neq \{e\}$ for some $j \neq i$, $1 \leq j \leq s$. Suppose that $\text{pr}_j(K_i) = G_j$. Then no simple component of K' other than K_i can project nontrivially to G_i and G_j . Consider the projections H' of H and K'_i of K_i to $G_i \times G_j$. Then $H'K'_i = G_i \times G_j$, and by reason of dimension we must have that $\text{pr}_i(H') = G_i$ and $\text{pr}_j(H') = G_j$. On the level of Lie algebras this says that we have a simple Lie algebra \mathfrak{g} and two subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 of $\mathfrak{g} \oplus \mathfrak{g}$ which project isomorphically to each \mathfrak{g} factor such that $\mathfrak{h}_1 + \mathfrak{h}_2 = \mathfrak{g} \oplus \mathfrak{g}$. But it follows from [Jac62a, Theorem 9] that $\mathfrak{h}_1 \cap \mathfrak{h}_2 \neq (0)$, a contradiction.

We may thus assume that $\dim \text{pr}_j(K_i) < \dim G_j$, hence $\dim H < \dim G_j$ as well. By Corollary 4.6 we have that $\text{pr}_j(K_i) \text{pr}_j(H) = G_j$ and that $\text{pr}_j(K')$ differs from $\text{pr}_j(K_i)$ by at most a factor of rank 1. Hence, with H' and K'_i as above, we again have $H'K'_i = G_i \times G_j$ which is not possible by reason of dimension. Hence we have that $\text{pr}_i(K') \neq G_i$ for $1 \leq i \leq s$. \square

Corollary 5.11. *Suppose that $1 \leq i \leq s$ and that $K_i \subset K'$ is a simple factor of rank at least two such that $\text{pr}_i(K_i) \neq \{e\}$. Then $\text{pr}_j(K_i) = \{e\}$ for $i \neq j$, $1 \leq j \leq s$.*

Proof. Suppose that $\text{pr}_j(K_i) \neq \{e\}$. If $\dim H < \dim G_i$ and $\dim H < \dim G_j$, then Corollary 4.6 shows that $\text{pr}_i(K_i)$ and $\text{pr}_j(K_i)$ differ from $\text{pr}_i(K')$ and $\text{pr}_j(K')$ by at most groups of rank 1, so that with H' and K'_i as above, we have $H'K'_i = G_i \times G_j$, which is not possible by reason of dimension. Thus $\text{pr}_i(H) = G_i$ (or $\text{pr}_j(H) = G_j$) which forces $\text{pr}_j(K_i) = G_j$ (or $\text{pr}_i(K_i) = G_i$), contradicting the lemma above. \square

Theorem 5.12. *Let H , s , r , $V = \bigoplus_i V_i \otimes W_i$, etc. be as above. Then one of the following occurs.*

- (1) $s = 1$ and $H = G_1$. Then $G'' = G_2 \times \cdots \times G_r$ and v corresponds to a subordination $(\bigoplus_i W_i^*, G'') \rightarrow (\bigoplus_i V_i, H)$ where W_i^* is sent to $V_i^{K'}$ for each i .
- (2) $s = 1$, $H \neq G_1$ and $G_1 = HK_1$ where $K_1 \subset G_1$ is a simple component of K . If $r > 1$, then $r = 2$, $G_2 = \text{SL}_2$ (up to a cover) and $(G_1, H, K_1 \times \text{SL}_2)$ is case 3.2 of Table 1. We have a subordination $(\bigoplus W_i^*, \text{SL}_2) \rightarrow (\bigoplus V_i, H)$ where the image of W_i^* is a subset of $V_i^{K_1}$ for each i .
- (3) $s = 2$, $\text{pr}_1(H) = G_1$, $\text{pr}_2(H) = G_2$ and there are subgroups K'_1, K'_2 of H such that (H, K'_1, K'_2) occurs in Table 1. We have $K = K_1 \times K_2$ where $K_i = \text{pr}_i(K'_i)$, $i = 1, 2$ and $G = HK$ (Example 5.6). If $r > 2$, then $r = 3$, $(H, K'_1, K'_2 \times \text{SL}_2)$ or $(H, K'_2, K'_1 \times \text{SL}_2)$ is entry 3.2 of Table 1, $G'' = \text{SL}_2$ and v corresponds to a subordination $(\bigoplus_i W_i^*, \text{SL}_2) \rightarrow (\bigoplus_i V_i, H)$ where W_i^* has image in $V_i^{K_1 \times K_2}$ for all i .
- (4) $r = s = 2$ where $\text{pr}_1(H) \neq G_1$ and $\text{pr}_2(H) = G_2$. Then we are in the case of Example 5.3, 5.4 or 5.5.

Proof. The cases where $s = 1$ are quite easy and we leave them to the reader. Now suppose that $s > 1$. Suppose that $\text{pr}_1(H) \neq G_1$ and that $\text{pr}_2(H) \neq G_2$. Then there are strongly semisimple

factors K_i of K' such that $\text{pr}_i(K_i)\text{pr}_i(H) = G_i$, $i = 1, 2$. By Lemma 5.10 and Corollary 5.11 the $\text{pr}_i(K_i)$ are proper subgroups of the G_i where $\text{pr}_2(K_1) = \text{pr}_1(K_2) = \{e\}$. Applying Proposition 5.7 we obtain that each of the G_i is isomorphic to H , a contradiction. Thus we may assume that $\text{pr}_1(H) = G_1$.

Let L' (resp. K_1) be the product of the strongly simple components of K' which map trivially (resp. nontrivially) to G_1 . Then $L' \subset G_2 \times \cdots \times G_s$. Since $G' = HK'$ and $\text{pr}_1(H) = G_1$, we must have that $G_2 \times \cdots \times G_s = H'L'$ where H' is the inverse image of $\text{pr}_1(K_1)$ in H projected to $G_2 \times \cdots \times G_s$. Since H' is a proper subgroup of H , $\text{pr}_i(H') \neq G_i$, $i = 2, \dots, s$. Since $\text{pr}_2(H')\text{pr}_2(L') = G_2$, we must have that H' is simple, by Table 1. By our argument above, we must have that $s = 2$.

Now suppose that $\text{pr}_1(H) = G_1$ and that $\text{pr}_2(H) = G_2$. Then by Lemma 5.10 and Corollary 5.11 we have that $H \times H \simeq G_1 \times G_2 = H(K_1 \times K_2)$ where the $K_i \subset G_i$ are images of simple subgroups K'_1 and K'_2 of K . Then $H = K'_1 K'_2$ so that (H, K'_1, K'_2) occurs in Table 1, as claimed. Suppose that $r > 2$. Then for $j > 2$, $K_j \subset G_1 \times G_2 \times G_j$ projects onto G_j and commutes with K_1 and K_2 . But the centralizer of $K_1 \times K_2$ in $G_1 \times G_2$ is trivial unless (H, K'_1, K'_2) or (H, K'_2, K'_1) is entry 3.1 of Table 1, in which case the centralizer is SL_2 . Thus $r = 3$ and there is a subordination as claimed.

Finally, suppose that $s = 2$ and that $\text{pr}_2(H) = G_2$ and $\text{pr}_1(H) \neq G_1$. Using Lemma 5.10 and Corollary 5.11 and the fact that $HK' = G_1 G_2$, we see that there are simple subgroups $K_i \subset G_i$, $i = 1, 2$, such that $H(K_1 K_2) = G_1 G_2$. We have entries $(G_1, \text{pr}_1(H), K_1)$ and $(\text{pr}_2(H), \text{pr}_2(L), K_2)$ in Table 1 where L is the preimage in H of $\text{pr}_1(H) \cap K_1$. Then H must be B_3 or of type A_{2n-1} . If $H = B_3$, then one easily sees that we are in Example 5.5 and that we cannot have $r > 2$. The remaining possibilities are that $H = A_{2n-1}$ and $G_1 = D_{2n}$ or B_3 giving Examples 5.3 and 5.4 where $r = 2$ is forced. \square

The theorem above gives one the possibilities for the semisimple part of the Levi factor of $\{g \in \text{GL}(V) \mid Gv = Hv\}$. Preferable would be a theorem which starts with a representation V of H and a generic $v \in V$ and tells you when v is almost semi-characteristic. In general, it is rather cumbersome to give such a theorem (for SL_2 see section 7). We content ourselves with working out the following example.

Example 5.13. Let $H = D_{2n+1}$, $n \geq 2$. Let $V = \bigoplus_{i=1}^k n_i V_i$ be the isotypic decomposition of the H -module V . Let $v = (v_1, \dots, v_k) \in V$ be generic. We find conditions which guarantee that v is almost semi-characteristic.

Each v_i is $(v_{i1}, \dots, v_{i,n_i})$ where v_{ij} lies in the j th copy of V_i , and the v_{ij} span a subspace $U_i \subset V_i$ of dimension n_i . In order to avoid case (1) of Theorem 5.12 we have to assume that the intersection of the stabilizers of the subspaces U_i in H contains no nontrivial semisimple group. Cases (2) and (4) do not apply, so we are left with case (3), where we have $G = H \times H$, $K_1 = B_{2n}$ and $K_2 = A_{2n}$. But then there is a copy of A_{2n-1} in D_{2n+1} which fixes our point. We have already ruled this out.

6. SEMISIMPLE GROUPS

We turn our attention to the case that $H \subset G$ where G and H are connected semisimple, V is an irreducible H -module, G acts almost faithfully on V and $Gv = Hv$ for some nonzero $v \in V$. Let G_1, \dots, G_k be the simple components of G . Then Theorem 4.10 tells us that $H = H_1 \cdots H_k$ where the H_i are semisimple and lie in G_i , $i = 1, \dots, k$. Note that no H_i is trivial, else G_i acts trivially on V . Thus if G_i has rank 1, then $G_i = H_i$. We have $V = V_1 \otimes \cdots \otimes V_k$ where V_i is an almost faithful irreducible representation of both G_i and H_i , $i = 1, \dots, k$.

6.1. Suppose that $G = HK$ where K is semisimple and G, H and V are as above. (Think of $K \subset G_v$.) Let pr_i denote the projection of G to G_i , $i = 1, \dots, k$. Let K' be a simple component

of K and set $I' := \{i \mid H_i \text{pr}_i(K') = G_i \text{ and } H_i \neq G_i\}$. We may assume that K contains no simple component of rank 1.

Proposition 6.2. *Let $G = HK$ as above. Let K', K'' be distinct simple components of K and let I' and I'' be as above. Then*

- (1) $I' \cap I'' = \emptyset$.
- (2) $|I'| \leq 2$. If $|I'| = 2$, then $\text{pr}_i(K') = G_i$ for some $i \in I'$.

Proof. For any $i \in I' \cap I''$, the (nontrivial) images of K' and K'' in G_i commute. This is clearly not possible if $\text{pr}_i(K')$ is G_i . If not, then we are in one of the entries of Table 1, and commutativity is not possible if $\text{pr}_i(K')$ is one of the groups occurring there. Hence (1) holds. Suppose that $i, j \in I'$, $i \neq j$ and $\text{pr}_i(K') \neq G_i$ and $\text{pr}_j(K') \neq G_j$. Then we must have that $H_j L = G_j$ where $L = \text{pr}_j(\text{pr}_i^{-1}(H_i) \cap K')$ is a proper subgroup of $\text{pr}_j(K')$ as in the last column of Table 1. But then, by inspection, we cannot have $H_j L = G_j$. If i, j and k are distinct elements of I' , then we can assume that $\text{pr}_i(K') = G_i$, and we derive a contradiction as before by considering the non-surjective projections of $\text{pr}_i^{-1}(H_i) \cap K'$ to G_j and G_k . Thus we have (2). \square

Theorem 6.3. *Suppose that $k = 2$ and that $H_1 \neq G_1$, $H_2 \neq G_2$ and $Gv = Hv$ for a nonzero $v \in V$. Then we are in one of the following cases.*

- (1) *There are tuples (G_i, V_i, H_i, K_i) in Table 2, $i = 1, 2$, and $v \in V_1^{K_1} \otimes V_2^{K_2}$.*
- (2) *The tuple (G_1, V_1, H_1, K_1) is entry (1) of Table 2, (G_2, V_2, H_2, K_2) is entry (3.3) (with the same k) and v generates the one-dimensional space of A_{2n-1} fixed vectors in $V_1 \otimes (V_2 | A_{2n-1})$.*
- (3) *The tuple (G_1, V_1, H_1, K_1) is entry (6.2) of Table 2, (G_2, V_2, H_2, K_2) is entry (6.3) (with $k = 1$) and v generates the one-dimensional space of D_8 fixed points in $V_1 \otimes V_2$.*

Proof. Let K_1 denote a maximal strongly semisimple subgroup of G_v . Suppose that $K_1 \subset G_1$. Then Table 1 implies that $H_1 K_1 = G_1$. Let K_2 be a strongly semisimple subgroup of G_v such that $\text{pr}_2(K_2) H_2 = G_2$. Then we must have $\text{pr}_1(K_2) = \{e\}$ (again by Table 1), hence $K_2 \subset G_2$ and we are in case (1). Thus we may suppose that any maximal strongly semisimple subgroup L of G_v lies diagonally in $G_1 \times G_2$. Since we are not in case (1), pr_2 restricted to L is almost faithful (and so is pr_1). By Table 1, L must be simple. It follows from Proposition 6.2 that we have two cases:

Case 1: $\text{pr}_1(L) = G_1$ and $\text{pr}_2(L) = K_2$ where (G_2, H_2, K_2) is in Table 1. Moreover, (G_2, H_2, K_2') is in Table 1, where $K_2' = \text{pr}_2(\text{pr}_1^{-1}(H_1) \cap L)$. Thus we are in the case $(G_2, H_2, K_2) = (D_{2n}, B_{2n-1}, A_{2n-1})$ and $(G_2, H_2, K_2') = (D_{2n}, B_{2n-1}, C_n)$, $n \geq 2$, where $H_1 \simeq C_n$. Then from Table 2 we see that $V_1 \simeq \varphi_1^k(A_{2n-1})$ (or its dual). From Table 2(3.3) we get possibility (2) of our theorem. From (3.1) we get nothing since A_{2n-1} has no fixed vectors in $V_1 \otimes V_2$. The possibilities (4.2) and (5.2) fail for the same reason. Hence we only get (2).

Case 2: Here we have that $\text{pr}_1(L) = G_1$ and $\text{pr}_2(L) = G_2$. Then $L = H_1' H_2'$ where $H_i' = \text{pr}_i^{-1}(H_i) \cap L$, $i = 1, 2$. Moreover, there are irreducible representations V_i of L whose restrictions to H_i' are irreducible, $i = 1, 2$. Table 2 tells us that we may have possibilities from entry (5.3) (and isomorphic entries), but then there are no D_4 -fixed points in $V_1 \otimes V_2$. Finally, from (6.2) and (6.3) we get possibility (3) above. \square

6.4. Let $H \subset \text{GL}(V)$ where H is semisimple connected and V is an irreducible H -module. Suppose that $v \in V$ is a nonzero orbit such that the connected semisimple part G of $\{g \in \text{GL}(V) \mid gHv = Hv\}$ is strictly larger than H . Let K denote the strongly semisimple part of G_v . We are then in the situation of 6.1. For each simple component K_j of K , let $I_j \subset \{1, \dots, k\}$ be as in 6.1. Set $I' = \cup_j I_j$, $V' = \otimes_{i \in I'} V_i$ and $V'' = \otimes_{i \notin I'} V_i$. Define G' and G'' analogously.

Then $G'' = H'' = \prod_{i \notin I'} H_i$. Let K' be the product of the K_j such that $K_j \subset G'$ and let K'' be the product of the other simple factors of K so we have $K = K'K''$. Via the projections to G' and H'' we have K'' -module structures on V' and V'' . Let $W' \subset V'$ be the minimal subspace such that $v \in W' \otimes V''$. Then $W' \subset (V')^{K'}$ and v is K'' -fixed.

Remark 6.5. It follows from Theorem 6.3 that each simple component of K' arises from an entry of Table 2 as in Theorem 6.3(1), is a group A_{2n-1} as in Theorem 6.3(2) or is the group D_8 in Theorem 6.3(3).

We restate the discussion in (6.4) as follows.

Theorem 6.6. *Let $v \in V$. If Hv is not semi-characteristic, then there are K' , K'' , etc. as in (6.4) and a minimal K'' -stable subspace $W' \subset (V')^{K'}$ such that $v \in W' \otimes V''$. If $K'' \neq \{e\}$, then there is a subordination $\alpha: ((W')^*, K'') \rightarrow (V'', H'')$.*

Now we would like to find some simple sufficient criteria for all generic v to be semi-characteristic. For this, we only need to avoid the case that $Gv = Hv$ where G_i differs from H_i for only one i . Then after renumbering we have that $G_1 \neq H_1$ and $G_i = H_i$ for $i > 1$. We have that $V'' = V_2 \otimes \cdots \otimes V_k$ and $H'' = H_2 \times \cdots \times H_k$. Note that H_1 may be any semisimple subgroup of H .

Proposition 6.7. *Let $G_1 \supset H_1$ be as above. Then one of the following occurs.*

- (1) *There is a subordination $\alpha: (V_1^*, G_1) \rightarrow (V'', H'')$ where K'' projects onto G_1 .*
- (2) *The tuple (V_1, G_1, H_1, K_1) occurs in Table 2 where K_1 is the projection of K'' to G_1 , and we have a subordination $(W_1^*, K_1) \rightarrow (V'', H'')$ where W_1 is minimal such that $v \in W_1 \otimes V''$.*
- (3) *The group K'' projects trivially to G_1 and the tuple (V_1, G_1, H_1, K_1) occurs in Table 2 for some K_1 where $V_1^{K_1} \neq (0)$.*

Proof. If $\text{pr}_1(K'') = \{e\}$, then we are in case (3). Suppose that $\text{pr}_1(K'')$ is nontrivial. Then it follows from Table 1 that $K' = \{e\}$. If the projection of K'' to G_1 is G_1 , then v corresponds to a subordination of V_1^* to V'' and we are in case (1). The only other possibility is that the projection of K'' is K_1 where (V_1, G_1, H_1, K_1) occurs in Table 2 and we are in case (2). \square

Example 6.8. Suppose that $k = 2$, $\dim V_2 \geq \dim V_1$ and that $H_2 = \text{SL}(V_2)$. Let $v \in V_1 \otimes V_2$ have maximal rank. Then case (1) applies. If (V_1, G_1, H_1, K_1) occurs in Table 2, let W_1 be any nontrivial K_1 -subspace of V_1 and let $v \in W_1 \otimes V_2$ have maximal rank. Then case (2) applies.

From Proposition 6.7 we get the following criterion for all nonzero orbits Hv to be semi-characteristic.

Corollary 6.9. *Let V be an irreducible H -module where H is semisimple. Write $H = H_1 \times \cdots \times H_k$ where the H_i are simple for $i > 1$, and let $V = V_1 \otimes \cdots \otimes V_k$ be the corresponding decomposition of V . Let V'' denote $V_2 \otimes \cdots \otimes V_k$ and set $H'' = H_2 \cdots H_k$. Suppose that none of the following occurs for any decomposition $H = H_1 \times \cdots \times H_k$.*

- (1) *There is a subordination $(V_1^*, H_1) \rightarrow (V'', H'')$ where $H_1 \neq \text{SL}(V_1)$.*
- (2) *There is a tuple (V_1, G_1, H_1, K_1) in Table 2 and a subordination $(W^*, K_1) \rightarrow (V'', H'')$ where $W \subset V_1$ is K_1 -stable.*
- (3) *There is a tuple (V_1, G_1, H_1, K_1) in Table 2 where $V_1^{K_1} \neq (0)$.*

Then every nonzero $v \in V$ is semi-characteristic.

Admittedly, the corollary is a little unwieldy, but in any concrete case it is quite easy to apply. We see what we can say in the case of isotropy representations of symmetric spaces.

Example 6.10. Let $H = A_5 \times A_1$ acting on $V = \varphi_3 \otimes \varphi_1$. This corresponds to the symmetric space of type EII (see [Hel78, Ch. X, Table V]). Let $0 \neq v \in V$ and let G be as usual with semisimple part G_s . If G_s contains H , then G_s cannot be simple (by Table 2), and if it is of the form $G_1 \times G_2$ where $G_1 \supset A_5$ and $G_1 = SL_2$, then it follows from Corollary 6.9 or Proposition 6.7 that $G_1 = A_5$. Hence v is semi-characteristic.

Example 6.11. Let $p \geq q \in \mathbb{N}$ where $p > 1$. Let $H = Sp_{2p} \times Sp_{2q}$ act in the natural way on $V = \mathbb{C}^{2p} \otimes \mathbb{C}^{2q}$. This corresponds to the symmetric space of type CII. Let $0 \neq v \in V$. Then one easily sees that the only possibility for a semisimple G_s containing H stabilizing Hv occurs in the case that v has rank 1, in which case $G_s = SL_{2p} \times SL_{2q}$. For $q > 1$ this corresponds to Theorem 6.3(1). If rank $v > 1$, then v is semi-characteristic.

Example 6.12. Let $p \geq q \geq 1$. Let H be the intersection of the block diagonal copy of $GL_p \times GL_q$ in GL_{p+q} with SL_{p+q} . Then H acts naturally on $V \oplus V^*$ where $V = \text{Hom}(\mathbb{C}^p, \mathbb{C}^q)$. This is an isotropy representation corresponding to the symmetric space of type AIII. First suppose that $q \geq 2$. Let $v = (x, x^*)$ where $x \in V$ and $x^* \in V^*$ are nonzero. Let $G = \{g \in GL(V \oplus V^*) \mid gHv = Hv\}^0$. Suppose that v is not semi-characteristic. Then \mathfrak{g} is an H -stable Lie subalgebra of $\text{Hom}(V \oplus V^*, V \oplus V^*)$ which properly contains \mathfrak{h} . If \mathfrak{g} projected to $\text{Hom}(V, V)$ or $\text{Hom}(V^*, V^*)$ is more than a central extension of \mathfrak{h} , then \mathfrak{g} has to contain \mathfrak{sl}_{p+q} . But there is no corresponding entry in Table 2. Thus we can suppose that \mathfrak{g} projects nontrivially to one of the irreducible components of

$$\text{Hom}(V, V^*) \simeq V^* \otimes V^* \simeq (S^2(\mathbb{C}^p) + \wedge^2(\mathbb{C}^p)) \otimes (S^2((\mathbb{C}^q)^*) + \wedge^2((\mathbb{C}^q)^*)).$$

Let us consider the case that \mathfrak{g} contains $\mathfrak{g}' := \wedge^2(\mathbb{C}^p) \otimes \wedge^2((\mathbb{C}^q)^*)$. Now x has normal form $\sum_{i=1}^k e_i^* \otimes f_i$ where e_1, \dots, e_p is a basis of \mathbb{C}^p , f_1, \dots, f_q is a basis of \mathbb{C}^q and $e_1^*, \dots, e_p^*, f_1^*, \dots, f_q^*$ denote the elements of the dual bases. Then $e_1 \wedge e_2 \otimes f_1^* \wedge f_2^*$ lies in \mathfrak{g}' and applied to x gives us $y^* = e_1 \otimes f_1^* + e_2 \otimes f_2^*$ if $k \geq 2$. The contraction of x and y^* (an H -invariant of $V \oplus V^*$) is not zero. Thus $(x, x^* + y^*)$ cannot be in the H -orbit of v , a contradiction. If $k = 1$, then acting by the reductive part of H_x we can bring x^* to the normal form

$$ce_1 \otimes f_1^* + e_1 \otimes f^* + e \otimes f_1^* + \sum_{i=2}^{\ell} e_i \otimes f_i^*$$

where $c \in \mathbb{C}$, $f^* \in \text{span}\{f_2^*, \dots, f_q^*\}$ and $e \in \text{span}\{e_2, \dots, e_p\}$. If $c \neq 0$, then acting by unipotent elements of H_x we can arrange that e and f^* are zero. Then $\ell + 1$ is an invariant of x^* (its rank) under the action of H_x . But we can change ℓ by adding elements of \mathfrak{g}' applied to x , again giving a contradiction. If $c = 0$ and $q > 2$, then one similarly sees that we can change the rank of x^* . If $c = 0$, $q = 2$ and e or $f^* \neq 0$, however, the G' -orbit of v is contained in the H_x -orbit of v and v is not semi-characteristic. The other three possible components of \mathfrak{g} give nothing new. Thus v is possibly not semi-characteristic only when $q = 2$, v is in the null cone and one of x and x^* has rank 1.

If $p = 1$ and $q = 1$ we have a torus action in which case $G = H$. If $q = 1$ and $p \geq 2$ we have the action of GL_p on $\mathbb{C}^p \oplus (\mathbb{C}^p)^*$. If Hv is not closed, then v is semi-characteristic. If Hv is closed, then $G \simeq SO_{2p}$ and v is not semi-characteristic.

Example 6.13. In general, one has a good chance to have points v which are not semi-characteristic in case your representation is reducible. One can calculate that this actually occurs for the following isotropy representations of symmetric spaces.

- (1) $(V, H) = (\mathbb{C}^p \otimes \mathbb{C}^2, SO_p \times SO_2)$, $p \geq 3$. This is of type BDI and of type CI for $p = 3$.
- (2) $(V, H) = (\wedge^2(\mathbb{C}^n) \oplus \wedge^2((\mathbb{C}^n)^*), GL_n)$, $3 \leq n \leq 5$. This is of type DIII.
- (3) $(V, H) = (\varphi_4 \otimes \nu_1 + \varphi_5 \otimes \nu_{-1}, D_5 \times \mathbb{C}^*)$. This is of type EIII. Here ν_j denotes the one-dimensional representation of \mathbb{C}^* of weight j .

(4) $(V, H) = (\varphi_1 \otimes \nu_1 + \varphi_5 \otimes \nu_{-1}, \mathbf{E}_6 \times \mathbb{C}^*)$. This is of type EVII.

Our discussion above establishes

Proposition 6.14. *Let (V, H) be the isotropy representation of an irreducible symmetric space. Then, with the exception of the adjoint representation of \mathbf{C}_n , $n \geq 2$ and the exceptions noted in Examples 6.11, 6.12 and 6.13, every orbit Hv , v generic, is semi-characteristic.*

The proposition applies to some of the questions of Raïs in [Raï07].

7. REPRESENTATIONS OF SL_2

We consider the case of H -modules V where $H := \mathrm{SL}_2$ and $V^H = 0$. We have a generic $v \in V$ and $G := \{g \in \mathrm{GL}(V) \mid gHv = Hv\}^0$ is not equal to H . We denote by R_n the H -module of binary forms of degree n . Then $R_n \simeq S^n(\mathbb{C}^2)$ has basis $x^n, x^{n-1}y, \dots, y^n$ where x, y are the usual basis of \mathbb{C}^2 and x^n is a highest weight vector. Let $N_G(H)$ denote the connected normalizer of H in G .

To determine G , we show that it suffices to determine $N_G(H)$ and \mathfrak{g}_u . We determine $N_G(H)$ in Theorem 7.4 below. We show that \mathfrak{g}_u is abelian and a multiplicity free H -module (Proposition 7.15). We give necessary and sufficient conditions for \mathfrak{g}_u to contain a copy of R_p , $p > 0$ (Theorem 7.27). We then find some simple conditions that guarantee that \mathfrak{g}_u is zero or the trivial H -module for every generic $v \in V$ (Corollary 7.29).

Lemma 7.1. *Let \tilde{G} be a Levi component of G containing H . Then $\tilde{G} \subset N_G(H)$.*

Proof. It follows from Theorem 4.3 that \tilde{G}_v contains the simple components of \tilde{G} of rank at least 2. These components are normalized by H so they fix the whole orbit Hv which spans V . Thus all the components of \tilde{G} have rank at most 1 and $\tilde{G} \subset N_G(H)$. \square

Corollary 7.2. *We have $\mathfrak{g} \simeq \tilde{\mathfrak{g}} \ltimes \mathfrak{g}_u$. Hence $G \neq N_G(H)$ if and only if \mathfrak{g}_u , as H -module, contains R_p for some $p > 0$. To determine G it suffices to determine $N_G(H)$ and \mathfrak{g}_u .*

We now consider the possibilities for $N_G(H)$.

Example 7.3. (1) Let \bar{H} be another copy of SL_2 and let $V_k = R_k \otimes \bar{R}_k$, $k \geq 1$, where \bar{R}_k is the \bar{H} -module of binary forms of degree k . Let $v_k \in V_k$ be a nonzero fixed vector of $\{(h, h)\} \subset H \times \bar{H}$. Then $Hv_k = (H \times \bar{H})v_k$ and v_k is generic. We can also take $V = V_{k_1} \oplus \dots \oplus V_{k_\ell}$ and $v = (v_{k_1}, \dots, v_{k_\ell})$ where $k_1 < \dots < k_\ell$. Then v is generic and $N_G(H) \supset H\bar{H}$.

(2) Let $V = \bigoplus_{k \in F} m_k R_k$ where $1 \leq m_k \leq k + 1$ for all k and F is a nonempty finite subset of \mathbb{N} . Let B and \bar{B} be the standard Borel subgroups of H and \bar{H} , respectively. Let v_k be the highest weight vector of the copy of $R_{2k+2-2m_k}$ in $V_k = R_k \otimes \bar{R}_k$ for the diagonal H -action. Then v_k lies in R_k tensored with the span of the weight vectors of \bar{R}_k of weight at least $k - 2m_k + 2$. Now v_k is an eigenvector for the diagonal copy of B , with weight $2k + 2 - 2m_k$. For $\bar{b} \in \bar{B}$, let $\chi(\bar{b})$ denote its upper left hand entry. Let \bar{b} act on \bar{R}_k as the tensor product of the usual action and the scalar action $\bar{b} \mapsto \chi(\bar{b})^{2m_k - 2k - 2}$. Then v_k is fixed by the diagonal in $B \times \bar{B}$. Assume that $m_k \geq 2$ for some k so that \bar{B} acts effectively. Set $v = \bigoplus_{k \in F} v_k$. Then v is generic and $N_G(H) \supset H\bar{B}$.

(3) Let $V = \bigoplus_{k \in F} m_k R_k$ as above. Let $v \in V$ be a generic vector whose projection $v_{k,j}$ to the j th copy of R_k is a weight vector. If the weight is not zero, then there is an obvious \mathbb{C}^* -action on this copy of R_k such that $v_{k,j}$ is fixed by the product of the standard torus in H and our external copy of \mathbb{C}^* . If v is not a sum of zero weight vectors we have $N_G(H) \supset H\mathbb{C}^*$.

Theorem 7.4. *Let $V = \bigoplus_{k \in F} m_k R_k$ be a representation of $H = \mathrm{SL}_2$ where $V^H = 0$. Let $v \in V$ be generic. If $N_G(H) \neq H$, then, up to the action of $\prod_{k \in F} \mathrm{GL}_{m_k}$, we are in one of the cases of Example 7.3. If $G \supset H\bar{H}$ as in Example 7.3(1), then $G = H\bar{H}$.*

Proof. We have $N_G(H) = HG'$ where G' is the identity component of the centralizer of H in G . The group G'_v fixes Hv , so it is trivial. Hence the Lie algebra of $G_v = \{hg' \mid hg'v = v\}$ projects onto \mathfrak{g}' and into \mathfrak{h} , so that G' is locally isomorphic to a quotient of a connected subgroup of H . Hence G' is locally isomorphic to a connected subgroup of H .

Case 1: $G' = \mathrm{SL}_2$ or SO_3 . Going to a cover, we can assume that $G' = \bar{H}$ so that G_v is isomorphic to the diagonal copy of H . Then V is a sum of representations $V_k = R_k \otimes S_k$ where S_k is a representation of \bar{H} of dimension at most $k+1$ and the projection of v to V_k is a fixed point of the diagonal action of H . Thus $S_k \simeq \bar{R}_k$ and v is as in Example 7.3(1). Suppose that $\mathfrak{g}_u \neq 0$. Then, as $(H \times \bar{H})$ -module, \mathfrak{g}_u cannot contain R_0 or \bar{R}_0 since the connected centralizer of H is \bar{H} and vice versa. Thus \mathfrak{g}_u contains a term $R_a \otimes \bar{R}_b$ where $ab \neq 0$. Hence, as H -module, \mathfrak{g}_u is not multiplicity free. But this contradicts Proposition 7.15 below. Hence $\mathfrak{g}_u = 0$ and $G = H\bar{H}$.
Case 2: $G' = \mathbb{C}^*$. Then $G_v \subset H \times \mathbb{C}^*$ is a diagonal torus and the fixed subspace of G_v on each isotypic component $m_k R_k$ of V is a sum of m_k distinct weight spaces of H . Thus we are in Example 7.3(3).

Case 3: $G' \supset \bar{U}$ and $G' \not\supset \bar{H}$ where $\bar{U} \subset \bar{H}$ is the standard maximal unipotent subgroup of our second copy \bar{H} of SL_2 . We have $V = \bigoplus_{k \in F} R_k \otimes S_k$ where S_k is a representation of \bar{U} . The isotropy group of v in $H \times \bar{U}$ can be taken to be the diagonal copy of U in $U \times \bar{U}$. Then v corresponds to a subordination $S_k^* \rightarrow R_k$, hence the image of S_k^* is a U -stable subspace of R_k and S_k is a \bar{B} -stable subspace of R_k . In fact, it is the span of $x^k, x^{k-1}y, \dots, x^{k-m_k+1}y^{m_k-1}$, and acting by elements of the various $\mathrm{GL}(m_k)$ we can arrange that v is as in Example 7.3(2). Hence $N_G(H) \simeq H\bar{B}$. \square

Corollary 7.5. *Let V be as above and $v \in V$ generic. Suppose that Example 7.3(1) does not apply so that $N_G(H) \neq H\bar{H}$. Then v is almost semi-characteristic.*

We now turn to the determination of \mathfrak{g}_u when it is not zero or a trivial H -module.

Proposition 7.6. *Let $v \in V$ be generic. Suppose that there is a copy of R_p in $\mathfrak{gl}(V)$, $p > 0$, which is a Lie subalgebra and acts nilpotently on V . Further suppose that $R_p(v) \subset \mathfrak{h}(v)$. Then $R_p \subset \mathfrak{g}_u$.*

Proof. Consider the action $\sigma: R_p \otimes V \rightarrow V$. Then for $h \in H$,

$$\sigma(R_p \otimes h(v)) = h\sigma(R_p \otimes v) \subset h\mathfrak{h}(v) = \mathfrak{h}(hv).$$

Hence Hv is open in $G_p v$ where G_p is the connected group with Lie algebra $\mathfrak{h} \ltimes R_p$. Thus $G_p \subset G$ and $R_p \subset \mathfrak{g}$. The projection of R_p to $\mathrm{Lie}(N_G(H))$ is trivial (by our classification of $N_G(H)$ and the fact that R_p is nilpotent). Hence $R_p \subset \mathfrak{g}_u$. \square

Remark 7.7. The proposition remains true if $p = 0$ as long as $G \neq H\bar{H}$ as in Example 7.3(1).

Example 7.8. Let $p, l > 0$. Let $v_{l+p} = x^{l+p}$ and let $v_l = a_0 x^l + a_1 x^{l-1}y$, $a_1 \neq 0$. Set $V = R_{l+p} + R_l$. Consider a nonzero equivariant map $\sigma: R_p \otimes R_{l+p} \rightarrow R_l$. Then $\sigma(x^i y^{p-i} \otimes v_{l+p})$ vanishes for $i > 0$ and $\sigma(y^p \otimes v_{l+p})$ is a nonzero multiple of x^l . Thus we may arrange that $\sigma(y^p \otimes x^{l+p}) = a_1 x^l$. If $A \in \mathfrak{sl}_2$ is $x\partial/\partial y$, then $\sigma(y^p \otimes v_{l+p}) = A(v_l)$. We may consider σ as an equivariant mapping of R_p to $\mathrm{Hom}(R_{l+p}, R_l)$. Then R_p applied to $v := v_{l+p} + v_l$ is the same as $\mathfrak{u}(v)$ where $\mathfrak{u} = \mathbb{C} \cdot A$. By Proposition 7.6, $R_p \subset \mathfrak{g}_u$. We can also have a copy of R_q in \mathfrak{g}_u , $q \neq p$, by adding R_{l+q} to V and adding v_{l+q} to v , where $v_{l+q} = x^{l+q}$.

We now try to pin down the structure of V and v . The situation can be quite complicated. First we need a lemma.

Lemma 7.9. *Let $\varphi: R_p \otimes R_n \rightarrow R_{p+n-2i}$ be equivariant and nonzero where $0 \leq i \leq \min\{p, n\}$. Then $\varphi(x^{p-j}y^j \otimes x^n) \neq 0$ for $i \leq j \leq p$.*

Proof. If $\varphi(x^{p-j}y^j \otimes x^n) = 0$, then the \mathfrak{sl}_2 -submodule W of $R_p \otimes R_n$ generated by $x^{p-j}y^j \otimes x^n$ lies in the kernel of φ . Applying $x\partial/\partial y \in \mathfrak{sl}_2$ repeatedly we may reduce to the case that $\varphi(x^{p-i}y^i \otimes x^n) = 0$. Suppose by induction that $x^{p-k}y^k \otimes x^{n-l}y^l$ lies in W for $k+l=i$ and $l \leq s$. Then applying $y\partial/\partial x$ followed by $x\partial/\partial y$ to $x^{p-k}y^k \otimes x^{n-s}y^s$ we obtain elements in W as well as $k(n-s)x^{p-k+1}y^{k-1} \otimes x^{n-s-1}y^{s+1}$. Thus W contains all the weight vectors of $R_p \otimes R_n$ of weight $p+n-2i$. This implies that $R_{p+n-2i} \subset W$, a contradiction. Thus $\varphi(x^{p-j}y^j \otimes x^n) \neq 0$. \square

Remark 7.10. By reversing the roles of x and y one has that $\varphi(x^jy^{p-j} \otimes y^n) \neq 0$ for $i \leq j \leq p$.

Corollary 7.11. *Let φ , etc. be as above where $p+n-2i \neq 0$. Let $w = x^n \in R_n$. Then $\dim \varphi(R_p \otimes w) \geq 2$ unless $i = p < n$ so that $\varphi(y^p \otimes w)$ is a highest weight vector of R_{n-p} .*

Set $W_0 = V$ and for $j > 0$ set $W_j = \mathfrak{g}_u(W_{j-1})$. Then W_j is a proper H -stable subspace of W_{j-1} for $j > 0$. Let k be the greatest integer j such that $W_j \neq 0$. Since \mathfrak{g}_u acts nontrivially on V , we must have $k > 0$. Let V_j be an H -complement to W_{j+1} in W_j for $0 \leq j \leq k$. Then $V = \bigoplus_j V_j$. Write $v = v_0 + v_1 + w_2$ where $v_i \in V_i$, $i = 1, 2$, and $w_2 \in W_2$. As before, let A denote $x\partial/\partial y \in \mathfrak{h}$.

Lemma 7.12. *Perhaps replacing v by hv for some $h \in H$ we have the following.*

- (1) *The vector v_0 is a sum of highest weight vectors.*
- (2) *The dimension of $\mathfrak{g}_u(v)$ is one with basis $A(v)$.*
- (3) *Suppose that $R_p \subset \mathfrak{g}_u$ where $p > 0$. Then for $p \geq i > 0$, $x^i y^{p-i} \in R_p$ annihilates v .*

Proof. Since \mathfrak{g}_u acts nontrivially on V and v is generic, there has to be a $C \in \mathfrak{g}_u$ such that $C(v) \in W_1$ and $C(v) \notin W_2$. Then there must be a $D \in \mathfrak{h}$ such that $D(v_0 + v_1) = C(v)$ modulo W_2 . Since D preserves V_0 and V_1 , we must have that $D(v_1) = C(v_0)$ modulo W_2 and that D annihilates v_0 . Up to the action of H , we may thus assume that v_0 is a sum of highest weight vectors or a sum of zero weight vectors. We assume the latter and derive a contradiction. Since \mathfrak{g}_u is H -stable, we may assume that C is a weight vector for the action of $\mathbb{C}^* \subset H$. Note that D generates the Lie algebra of $\mathbb{C}^* \subset H$. If C has weight zero, then so does $C(v) + W_2 = C(v_0) + W_2$ and we cannot have that $C(v) = D(v)$ modulo W_2 . Thus C has weight j for some $j \neq 0$ so that $C(v_0) + W_2 = D(v_1) + W_2$ also has weight j . Hence $v_1 = v'_1 + v''_1$ where $v'_1 + W_2 = C(v_0) + W_2$ and v''_1 has weight j while v'_1 has weight 0. Now let Z be the two-dimensional vector space generated by v_0 and v'_1 , all modulo W_2 . The groups generated by $\exp(tD)$ and $\exp(tC)$, $t \in \mathbb{C}$, act on Z and the orbits of (v_0, v'_1) are the same. But $\exp(tC)(v_0, v'_1)$ contains the point $(v_0, 0)$ while $\exp(tD)(v_0, v'_1)$ clearly does not. Hence we have (1), i.e., v_0 is a sum of highest weight vectors. Moreover, $\mathfrak{g}_u(v) + W_2$ is one-dimensional and generated by $A(v_1) + W_2$.

Let $C \in \mathfrak{g}_u$ as above. Then $C(v) = D(v)$ for some $D \in \mathfrak{h}_{v_0}$, where D is a multiple of A . Hence we have (2). Finally, suppose that $R_p \subset \mathfrak{g}_u$ where $p > 0$. By Corollary 7.11, for $i > 0$, $x^i y^{p-i}$ annihilates v , modulo W_2 , while y^p sends v to a multiple of $A(v)$, modulo W_2 . If $x^i y^{p-i}$ acts nontrivially on v it follows that $\dim \mathfrak{g}_u(v) > 1$. Hence we have (3). \square

Let $\sigma: R_p \otimes V \rightarrow V$ be the action of some $R_p \subset \mathfrak{g}_u$ where $p \geq 0$. Let $\mu: V \rightarrow V$ be the action of y^p via σ . We may assume that $\mu(v) = A(v)$.

Corollary 7.13. (1) *For all $j \geq 1$, $\mu^j(v) = A^j(v)$.*
 (2) *If $p > 0$, then for all $1 \leq i \leq p$, $j \geq 0$, $\sigma(x^i y^{p-i} \otimes A^j(v)) = 0$.*

Proof. Suppose that $p > 0$. We prove (1) and (2) simultaneously by induction on j . Assume that $\mu^j(v) = A^j(v)$ for $1 \leq j \leq m$ and that $\sigma(x^i y^{p-i} \otimes A^j(v)) = 0$ for $0 \leq j < m$, $i > 0$. We

certainly have the case that $m = 1$. Apply A to the equation $\sigma(y^p \otimes A^{m-1}(v)) = A^m(v)$. Since σ is equivariant, one obtains that

$$\sigma(pxy^{p-1} \otimes A^{m-1}(v)) + \sigma(y^p \otimes A^m(v)) = A^{m+1}(v).$$

Since the first term above is zero, we have that $\mu(A^m(v)) = A^{m+1}v$ so that, by induction, we have $\mu^{m+1}(v) = A^{m+1}(v)$. Now apply A to the equation $\sigma(x^i y^{p-i} \otimes A^{m-1}(v)) = 0$. One obtains that

$$\sigma((p-i)x^{i+1}y^{p-i-1} \otimes A^{m-1}(v)) + \sigma(x^i y^{p-i} \otimes A^m(v)) = 0$$

so that $\sigma(x^i y^{p-i} \otimes A^m(v)) = 0$. This completes the induction. In case $p = 0$, A commutes with the generator of R_0 , so that (1) is immediate. \square

Remark 7.14. Suppose that $p > 0$ and that we have (1) above. Then applying A to the equations of (1) and using induction we obtain (2).

Proposition 7.15. *The Lie algebra \mathfrak{g}_u is abelian and as H -module is multiplicity free.*

Proof. Suppose that we have copies of R_p and R_q in \mathfrak{g}_u where we allow $p = q$ (in which case we have two copies of R_p). If $[R_p, R_q] \neq 0$, then we have a copy of some R_s in \mathfrak{g}_u which maps V to W_2 . Thus $R_s(v) \neq 0$ while $R_s(v) \in W_2$. This implies, as in the proof of Lemma 7.12, that $\mathfrak{g}_u(v)$ has dimension greater than one, a contradiction. Hence \mathfrak{g}_u is abelian. If R_p has multiplicity two or more, then it follows from Lemma 7.12 that there is a copy of R_p which sends v to 0 implying that this copy of R_p acts trivially on V , a contradiction. Hence \mathfrak{g}_u is multiplicity free. \square

Proposition 7.16. *For all $i \geq 0$, $A^i(v)$ is generic in W_i .*

Proof. Since v is generic in V and \mathfrak{g}_u is H -stable, W_1 is generated by the H -orbit of $\mathfrak{g}_u(v)$. Hence Av is generic in W_1 . Then the same argument shows that the H -orbit of $A^2(v)$ spans W_2 , etc. \square

We say that a vector $w \in R_l$ has *height k* if $w = a_0 x^l + \cdots + a_k x^{l-k} y^k$ where $a_k \neq 0$. A vector in $Z := \sum_i m_i R_i$ has *height at least k* (resp. *height at most k*) if it is generic in Z and when written as a sum $\sum_i v_{i,1} + \cdots + v_{i,m_i}$ where $v_{i,j}$ is in the j th copy of R_i , each $v_{i,j}$ has height at least k (resp. at most k).

Proposition 7.17. *The H -modules V_i are multiplicity free.*

Proof. The vector $A^j v$ is generic in W_j , $j \geq 0$, and the projection of $A^j v$ to any R_l in W_j cannot be zero. Thus the projection of v to any $R_l \subset W_j$ has height at least j . We have $v + W_j = v_0 + v_1 + \cdots + v_{j-1} + W_j$ where $A^j v \in W_j$. It follows that $A^j v_i = 0$ for $i < j$, hence any v_i is a sum of vectors of height at most i . Since $v_j \in W_j$ it is a sum of vectors of height at least j . Thus $A^j v_j$ is a sum of highest weight vectors and it is generic in W_j . Hence any R_l can occur in W_j with multiplicity at most one. \square

Write $v = v_0 + \cdots + v_k$ where $v_i \in V_i$. Then each v_i is a sum $\sum_{l \in F_i} v_{i,l}$ where $F_i \subset \mathbb{N}$ and $v_{i,l}$ lies in the copy of $R_l \subset V_i$.

Corollary 7.18. *Each vector $v_{i,l}$ has height i .*

Lemma 7.19. *Let $\varphi: R_p \otimes R_m \rightarrow R_l$ be equivariant and nonzero.*

- (1) *Necessarily $m = l + p - 2i$ for some i with $0 \leq i \leq \min\{l, p\}$.*
- (2) *Suppose that $w \in R_m$ has height $n \leq l - i$. Then $\varphi(y^p \otimes w)$ has height $n + i$.*

Proof. Since representations of H are self-dual, R_l appears in $R_p \otimes R_m$ if and only if R_m appears in $R_p \otimes R_l$. Then Clebsch-Gordan implies (1). Now consider $z := \varphi(y^p \otimes x^{m-n} y^n)$ where $m = l + p - 2i$. Then Remark 7.10 shows that $z \neq 0$ if the weight of $y^p \otimes x^{m-n} y^n$ is at least $-l$. This is equivalent to $n \leq l - i$, hence we have (2). \square

As above, we have $v_i = \sum_{l \in F_i} v_{i,l}$ where $v_{i,l} \in R_l \subset V_i$. For any $s \geq 0$, we have $W_1 = V_1 \oplus V_2 \oplus \cdots \oplus V_{s+1} \oplus W_{s+2}$, hence we have an H -equivariant projection of W_1 to V_{s+1} . Let τ denote σ on $R_p \otimes (V_0 + \cdots + V_s)$ followed by projection onto $R_l \subset V_{s+1}$. Since we have $\sigma(y^p \otimes A^r(v)) = A^{r+1}(v)$, $r \geq 0$, for every $v_{s+1,l} \in R_l \subset V_{s+1}$, $A^{r+1}(v_{s+1,l})$ must be a multiple of $\tau(y^p \otimes A^r(v_0 + \cdots + v_s))$ for $r \geq 0$. Note that τ vanishes on $R_p \otimes v_{i,t}$ unless $t = l + p - 2j$ where $0 \leq j \leq \min\{p, s\}$.

Proposition 7.20. *Let s and l be as above. Let $R_{l+p-2j} \subset V_i$, $j \leq \min\{p, s\}$. If $i + j > s$, then $\tau(R_p \otimes R_{l+p-2j}) = 0$.*

Proof. Consider the pairs (i', j') where $0 \leq i' \leq s$, $0 \leq j' \leq \min\{p, s\}$, $i' + j' > s$ and $v_{i', l+p-2j'} \neq 0$. Assume that i is the maximal i' that occurs and that j is the maximal j' that occurs in a pair (i, j') . Consider $A^i(v_{i, l+p-2j})$. It is a highest weight vector of weight $l + p - 2j$. Suppose that $\tau(y^p \otimes A^i(v_{i, l+p-2j}))$ is nonzero. Then it has height $j > s - i$. Moreover, by the choice of i and j , $\tau(y^p \otimes A^i(v_{i, l+p-2j}))$ is the nonzero $\tau(y^p \otimes A^i(v_{i, l+p-2j}))$ of largest height (equivalently, of lowest weight). But $\tau(y^p \otimes A^i(v)) = A^{i+1}(v_{s+1, l})$ where $A^{i+1}(v_{s+1, l})$ has height $s - i$. Thus $\tau(y^p \otimes A^i(v_{i, l+p-2j}))$ must be zero. Now for $0 < m \leq p$ we have that $\sigma(x^m y^{p-m} \otimes A^i(v)) = 0$. Again, by height considerations, one sees that $\tau(x^m y^{p-m} \otimes A^i(v_{i, l+p-2j}))$ must vanish. Hence $\tau(R_p \otimes A^i(v_{i, l+p-2j})) = 0$ which shows that $\tau(R_p \otimes R_{l+p-2j}) = 0$. Now the proof can be completed by downward induction on i and j . \square

For $0 \leq j \leq \min\{p, s\}$ and $R_{l+p-2j} \subset V_i$, the restriction of τ to $R_p \otimes R_{l+p-2j}$ is a multiple $t_{i, s-j} \tau_{s-j}$ of τ_{s-j} where $\tau_{s-j}: R_p \otimes R_{l+p-2j} \rightarrow R_l$ is equivariant and normalized so that $\tau_{s-j}(y^p \otimes x^{l+p-2j}) = x^{l-j} y^j$. We have that

$$(7.20.1) \quad \sum_{j=0}^{\min\{p, s\}} \sum_{i=0}^{s-j} t_{i, s-j} \tau_{s-j}(y^p \otimes A^r(v_{i, l+p-2j})) = A^{r+1}(v_{s+1, l})$$

for all $r \geq 0$ where we set $v_{i, q} = 0$ if $q \notin F_i$.

Let a_i and $b_{i, m}^{s-j}$ be scalars such that $v_{s+1, l} = a_0 x^l + \cdots + a_{s+1} x^{l-s-1} y^{s+1}$ and $v_{i, l+p-2j} = b_{i, 0}^{s-j} x^{l+p-2j} + \cdots + b_{i, i}^{s-j} x^{l+p-2j-i} y^i$ whenever $l + p - 2j \in F_i$ and $0 \leq j \leq \min\{p, s\}$. We use t_j and b_j as shorthand for $t_{j, j}$ and $b_{j, j}^j$, respectively. For now assume that $l + p - 2j \in F_{s-j}$, $j = 0, \dots, \min\{p, s\}$.

Theorem 7.21. *For $m = 0, \dots, s$, consider the equation (7.20.1) with $r = m$ in weight $l - 2s + 2m$. This gives us $s + 1$ equations in the unknowns $t_{s-j} b_{s-j}$ for $0 \leq j \leq \min\{p, s\}$. The unique solutions are*

$$t_{s-j} b_{s-j} = \frac{\binom{s}{j} \binom{p}{j}}{\binom{l+p-j+1}{j}} (s+1) a_{s+1}, \quad 0 \leq j \leq \min\{p, s\}.$$

Proof. First assume that $p \geq s$. Since $\tau_{s-j}(y^p \otimes x^{l+p-2j}) = x^{l-j} y^j$, it follows that

$$\tau_{s-j}(y^p \otimes x^{l+p-2j-k} y^k) = \frac{\binom{l-j}{k}}{\binom{l+p-2j}{k}} x^{l-j-k} y^{j+k}, \quad k \leq l-j.$$

Now the 0th equation ($m = 0$) is

$$t_0 b_0 + t_1 b_1 \frac{l-s+1}{l+p-2s+2} + \cdots + t_j b_j \frac{\binom{l-s+j}{j}}{\binom{l+p-2s+2j}{j}} + \cdots + t_s b_s \frac{\binom{l}{s}}{\binom{l+p}{s}} = (s+1) a_{s+1}.$$

For $m = 1$ the equation is

$$t_1 b_1 + 2t_2 b_2 \frac{l-s+2}{l+p-2s+4} + \cdots + j t_j b_j \frac{\binom{l-s+j}{j-1}}{\binom{l+p-2s+2j}{j-1}} + \cdots + s t_s b_s \frac{\binom{l}{s-1}}{\binom{l+p}{s-1}} = s(s+1) a_{s+1}$$

and the m th equation is

$$m!t_m b_m + \cdots + j!/(j-m)!t_j b_j \frac{\binom{l-s+j}{j-m}}{\binom{l+p-2s+2j}{j-m}} + \cdots + \frac{s!}{(s-m)!}t_s b_s \frac{\binom{l}{s-m}}{\binom{l+p}{s-m}} = \frac{(s+1)!}{(s-m)!}a_{s+1}.$$

Thus our system of equations is equivalent to

$$(7.21.1) \quad \sum_{j=0}^s \binom{j}{m} c_j \frac{\binom{l-s+j}{j-m}}{\binom{l+p-2s+2j}{j-m}} = \binom{s}{m}, \quad m = 0, \dots, s$$

where $c_j = t_j b_j / ((s+1)a_{s+1})$. Since the equations are in triangular form, there is a unique solution. Now the theorem will be proved if we can show that a solution to (7.21.1) is

$$c_{s-j} = \frac{\binom{s}{j} \binom{p}{j}}{\binom{l+p-j+1}{j}}.$$

But one can prove this using the WZ method [WZ90]. (See [Tef10] for a brief introduction.) We used the implementation of the WZ method in MAPLE.

Now suppose that $p < s$. We may still consider the system of equations (7.21.1). The solutions remain the same, but note that for $j > p$, the formula for c_{s-j} gives zero. Hence the theorem is true even when $p < s$. \square

Remark 7.22. Let $0 \leq j \leq \min\{p, s\}$. We assumed that R_{l+p-2j} occurred in V_{s-j} . But the equations force $t_{s-j}b_{s-j}$ to be nonzero. Thus, in fact, R_{l+p-2j} must occur in V_{s-j} for there to be a solution of (7.20.1) for all $r \geq 0$.

Since we are guaranteed to have vectors $v_{s-j, l+p-2j}$ in our solution of (7.20.1), what role do the vectors $v_{i, l+p-2j}$ play for $i < s-j$? It is easy to see that the term involving $v_{i, l+p-2j}$ may be eliminated if we change $v_{s-j, l+p-2j}$ to $v_{s-j, l+p-2j} + \frac{t_{i, s-j}}{t_{s-j}} v_{i, l+p-2j}$. Let us say that $v'' = \sum_j v'_{s-j, l+p-2j}$ is obtained from $v' = \sum_j v_{s-j, l+p-2j}$ by an *admissible modification* if each $v'_{s-j, l+p-2j}$ differs from $v_{s-j, l+p-2j}$ by a linear combination of the $v_{i, l+p-2j}$ for $i < s-j$. Thus we have the following

Remark 7.23. We have a solution of (7.20.1) if and only if, up to an admissible modification of the $v_{s-j, l+p-2j}$, we have a solution of

$$(7.23.1) \quad \sum_{j=0}^{\min\{p, s\}} t_{s-j} \tau_{s-j} (y^p \otimes A^r(v_{s-j, l+p-2j})) = A^{r+1}(v_{s+1, l}), \quad r \geq 0.$$

Proposition 7.24. Let $v_{s+1, l} \in V_{s+1}$ and $v_{s-j, l+p-2j} \in V_{s-j}$ have coefficients a_j and $b_{i, m}^{s-j}$ as above. Fix the b_{s-j} , $0 \leq j \leq \min\{p, s\}$. Then there are unique values of the t_{s-j} and $b_{s-j, m}^{s-j}$ for $m < s-j$ such that there is a solution of (7.23.1).

Proof. We know that the t_{s-j} are uniquely determined. We only need to show that the $b_{s-j, m}^{s-j}$ for $m < s-j$ are unique satisfying (7.23.1). This is easy because of the triangular form of the equations. For $r = 0$, the equation in weight l reads $t_s b_{s, 0}^s = a_1$. For arbitrary $r \leq s$, the equation in weight l is $r!t_s b_{s, r}^s = (r+1)!a_{r+1}$. For $r = s$ this is one of the equations we considered in Theorem 7.21 and we have $b_{s, r}^s = \frac{r+1}{t_s} a_{r+1}$ for $r < s$. Now suppose that we have determined the $b_{s-q, j}^{s-q}$ for $0 \leq q < m$. Consider (7.23.1) in weight $l - 2m$ with $r = 0$. It gives an expression for $t_{s-m} b_{s-m, 0}^{s-m}$ in terms of the a_i and $b_{s', s'-j}^{s'}$ for $s' > s-m$. Thus we may solve for $b_{s-m, 0}^{s-m}$. For $0 < r \leq s-m$ we obtain an equation that we can solve for $b_{s-m, r}^{s-m}$. The equation that we get for b_{s-m} is one of the equations that we considered in Theorem 7.21. Hence given a_1, \dots, a_{s+1} and the b_{s-j} , there are unique t_{s-j} and $b_{s-j, m}^{s-j}$ solving (7.23.1). \square

Remark 7.25. Suppose that $l + p - 2j \in F_i$ for all $i + j \leq s$, $0 \leq j \leq \min\{p, s\}$. Then we may modify the $v_{s-j, l+p-2j}$ admissibly so that the $b_{s-j, m}^{s-j}$, $m < s - j$, are arbitrary. Hence there are t_{s-j} giving solutions of (7.23.1) (after admissible modifications) and giving solutions of (7.20.1) (without changing any vectors).

Let us formulate the conditions that need to be satisfied to have $R_p \subset \mathfrak{g}_u$, $p > 0$.

Definition 7.26. Let $v \in V$ be generic. We say that v satisfies $(*_p)$ if

- (1) We have a decomposition $V = \bigoplus_{i=0}^k V_i$ where the V_i are multiplicity free H -modules. Let $F_i \subset \mathbb{N}$ such that $V_i = \bigoplus_{l \in F_i} R_l$.
- (2) Possibly replacing v by hv for some $h \in H$, we have that $v = \sum_i \sum_{l \in F_i} v_{i, l}$ where $v_{i, l} \in R_l \subset V_i$ has height i .
- (3) For every $v_{s+1, l} \in V_{s+1}$, $s \geq 0$, we have that $l + p - 2j \in F_{s-j}$ for $0 \leq j \leq \min\{p, s\}$. Let the t_{s-j} be given by Theorem 7.21. Then the vectors $v_{s-j, l+p-2j}$, perhaps after admissible modification, are solutions of (7.23.1).

Theorem 7.27. Let $v \in V$ be generic. Then $R_p \subset \mathfrak{g}_u$, $p > 0$, if and only if v satisfies $(*_p)$.

Proof. We have shown that $R_p \subset \mathfrak{g}_u$ implies that $(*_p)$ holds. Conversely, if $(*_p)$ holds, then we have constants $t_{i, s-j}$ such that (7.20.1) is satisfied. Let τ denote the corresponding map

$$R_p \otimes \left(\bigoplus_{j=0}^{\min\{p, s\}} \bigoplus_{i=0}^{s-j} \bigoplus_{l+p-2j \in F_i} R_{l+p-2j} \subset V_i \right) \rightarrow R_l \subset V_{s+1}.$$

The various mappings τ combine to give us an equivariant map $\sigma: R_p \otimes V \rightarrow V$. It follows from Remark 7.14 that $\sigma(x^i y^{p-i} \otimes v) = 0$ for $i > 0$. By construction, $\sigma(R_p \otimes v)$ is one-dimensional and generated by $\sigma(y^p \otimes v) = A(v)$. If S denotes the copy of $R_p \subset \text{End}(V)$ corresponding to σ , then $[S, S](v) = 0$ which implies that $[S, S]$ acts trivially on V , i.e., $[S, S] = 0$. By construction, S consists of nilpotent transformations. Now by Proposition 7.6 we have $S \subset \mathfrak{g}_u$. \square

Corollary 7.28. Suppose that there is a generic $v \in V$ such that \mathfrak{g}_u is not zero or the trivial H -module. Then there are subsets $F_0, \dots, F_k \subset \mathbb{N}$ such that $V = \bigoplus_{i=0}^k \bigoplus_{l \in F_i} R_l$ and $p > 0$ such that for every $l \in F_{s+1}$, $s \geq 0$ we have $l + p - 2j \in F_{s-j}$ for $0 \leq j \leq \min\{p, s\}$.

Corollary 7.29. The group G normalizes H if V does not satisfy the condition of Corollary 7.28. In particular, G normalizes H in the following cases.

- (1) V is an isotypic H -module.
- (2) The multiplicity of R_l is at least two, where l is maximal such that $R_l \subset V$.

Proof. Part (1) is clear. In case (2), there has to be a vector $v_{s+1, l}$ where $s \geq 0$. Thus we must have $R_{p+l} \subset V_s$, which obviously fails. \square

Using Remark 7.25 it is clear that one can have extremely complicated situations where $R_p \subset \mathfrak{g}_u$. Here is a modestly complicated case.

Example 7.30. Let $V_0 = R_{l+p-2} \oplus R_{l+2p}$, $V_1 = R_{l+p}$ and $V_2 = R_l$, where $l > 1$, $p > 0$. Let $v = v_{0, l+p-2} + v_{0, l+2p} + v_{1, l+p} + v_{2, l} \in V = V_0 \oplus V_1 \oplus V_2$ where the $v_{r, s}$ are of height r in $R_s \subset V_r$. Then by Remark 7.25, \mathfrak{g}_u contains a copy of R_p . Here we have that $\sigma(y^p \otimes v_{0, l+2p}) = A(v_{1, l+p})$ and $\sigma(y^p \otimes A^r(v_{0, l+p-2} + v_{1, l+p})) = A^{r+1}(v_{2, l})$, $r = 0, 1$. If we add a copy of R_l to V_1 and a copy of R_l to V_0 (assume $p \neq 2$) with corresponding components $v_{1, l}$ and $v_{0, l}$ in v , then we also have a copy of R_0 in \mathfrak{g}_u . If $p = 2$, we already have $R_l \subset V_0$ and we only have to add $R_l \subset V_1$ and $v_{1, l}$.

Example 7.31. Suppose that $V = 2R_l \oplus R_{l-1} \oplus R_{l+1}$ where $l \geq 2$. Then it is possible to have a generic $v \in V$ such that $R_1 \subset \mathfrak{g}_u$. However, one can check that this is not possible if we increase the multiplicity of R_l to 3.

8. APPENDIX

Here we establish the branching rules which are used in Table 2 and the calculation of V^K in cases 6.3 and 6.4. Recall that if φ is a G -module, then $\mathcal{S}(\varphi) = \bigoplus_k \mathcal{S}^k(\varphi)$ where $\mathcal{S}^k(\varphi)$ denotes the subspace of $S^k(\varphi)$ obtained using Cartan multiplication of the irreducible subrepresentations of φ .

Let $n = 2m + 1$, $m \geq 1$. Let X denote the cone in $\varphi_{n-1}(\mathbf{D}_n)$ which is the closure of the orbit of a highest weight vector. Consider the action of $\mathrm{SL}_n \times \mathbb{C}^*$ on $\mathcal{O}(X)$ where $\varphi_1(\mathbf{D}_n) = \mathbb{C}^{2n} = \varphi_1(\mathrm{SL}_n) \oplus \varphi_{n-1}(\mathrm{SL}_n) = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$. Here \mathbb{C}^n is the span of the positive weight vectors of \mathbf{D}_n , \mathbb{C}^* acts on \mathbb{C}^n with weight 2 and on $(\mathbb{C}^n)^*$ with weight -2 . As $\mathrm{SL}_n \times \mathbb{C}^*$ representation, $\varphi_{n-1}(\mathbf{D}_n)$ is $\nu_n \oplus \varphi_{n-2}(\mathrm{SL}_n) \otimes \nu_{n-4} + \varphi_{n-4}(\mathrm{SL}_n) \otimes \nu_{n-8} + \cdots + \varphi_1(\mathrm{SL}_n) \otimes \nu_{-n+2}$ and $\varphi_n(\mathbf{D}_n) = \varphi_{n-1}(\mathbf{D}_n)^* = \varphi_{n-1}(\mathrm{SL}_n) \otimes \nu_{n-2} + \cdots + \varphi_2(\mathrm{SL}_n) \otimes \nu_{-n+4} + \nu_{-n}$.

Theorem 8.1. *Let X be the closure of the highest weight orbit in $\varphi_{n-1}(\mathbf{D}_n)$. Then, as $(\mathrm{SL}_n \times \mathbb{C}^*)$ -module, $\mathcal{O}(X) = \mathcal{S}(\varphi_n(\mathbf{D}_n))$.*

Proof. It is well-known that X is normal, and every point of X except the origin is smooth. For $x \in \nu_n$, $x \neq 0$, x is a smooth point of X , x is a fixed point of SL_n and the slice representation of SL_n at x is $\theta_1 + \varphi_{n-2}$. Since φ_{n-2} has no invariants, $\mathcal{O}(X)^{\mathrm{SL}_n}$ is generated by a coordinate function z on ν_n . Then z is not a zero divisor in $\mathcal{O}(X)$, hence $\mathcal{O}(X)$ is a free $\mathbb{C}[z]$ -module. By Luna's slice theorem [Lun73], $\mathcal{O}(X)$ is a free $\mathbb{C}[z]$ -module on $\mathcal{O}(\varphi_{n-2})$. But $\mathcal{O}(\varphi_{n-2}) = S(\varphi_2)$ is just the sum of all representations of the form $\varphi_2^{a_2} \cdot \varphi_4^{a_4} \cdots \varphi_{n-1}^{a_{n-1}}$ each with multiplicity one. It follows that the products of the highest weight vectors of the restriction of $\varphi_{n-1}(\mathbf{D}_n)^*$ to SL_n freely generate the highest weights of $\mathcal{O}(X)$ as an SL_n -module and as an $(\mathrm{SL}_n \times \mathbb{C}^*)$ -module. \square

Now suppose the $n = 2m$, $m \geq 2$. Let X denote the closure of the orbit of a highest weight vector of $\varphi_{n-1}(\mathbf{D}_n)$. Consider the action of $\mathrm{SL}_n \times \mathbb{C}^* \subset \mathbf{D}_n$ such that $\varphi_1(\mathbf{D}_n)$ becomes $\mathbb{C}^n \otimes \nu_1 \oplus (\mathbb{C}^n)^* \otimes \nu_{-1}$. Effectively, we have the action of GL_n . Then $\varphi_{n-1}(\mathbf{D}_n)$, as a GL_n -module, is $\nu_m + \varphi_{n-2} \otimes \nu_{m-2} + \cdots + \nu_{-m}$.

Theorem 8.2. *Let $n = 2m$, $m \geq 2$ and let X be the closure of the highest weight orbit in $\varphi_{n-1}(\mathbf{D}_n)$. Then, as GL_n -module, $\mathcal{O}(X) = \mathcal{S}(\varphi_{n-1}(\mathbf{D}_n))$.*

Proof. Let z_{\pm} be coordinate functions on the copies of $\nu_{\pm m}$ in $\varphi_{n-1}(\mathbf{D}_n)$. As above, one computes that there is a slice representation $(\varphi_{2m-2} + \theta_1, \mathrm{SL}_{2m})$ for the action of SL_{2m} on X . The slice representation has a quotient of dimension two and principal isotropy group \mathbb{C}_m . It follows that the GL_n -invariants have dimension 1, hence they must be generated by $z_+ z_-$. Moreover, the only way that the trivial SL_n -representation can occur in $\mathbb{C}[\varphi_{n-2} \otimes \nu_{m-2} + \cdots + \varphi_2 \otimes \nu_{-m+2}]$ is in products whose \mathbb{C}^* -weight is a multiple of $\pm m$ (just count boxes in Young diagrams). Since GL_n is spherical in \mathbf{D}_n , each ν_{km} , $k \in \mathbb{Z}$, occurs once in the free $\mathbb{C}[z_+ z_-]$ -module $\mathcal{O}(X)$. Thus the SL_n -invariants must be the polynomial ring $\mathbb{C}[z_+, z_-]$ and $\mathcal{O}(X)$ is free over $\mathbb{C}[z_+, z_-]$. For the corresponding map $X \rightarrow \mathbb{C}^2$, the general fiber is SL_n/\mathbb{C}_m , which gives that the only SL_n -representations that occur are $\varphi_2^{a_2} \cdots \varphi_{n-2}^{a_{n-2}}$ for $a_2, \dots, a_{n-2} \geq 0$, each with multiplicity one. It follows that $\mathcal{O}(X) = \mathcal{S}(\varphi_{n-1}(\mathbf{D}_n))$. \square

Finally, we consider the case where X is the closure of the highest weight vector in $\varphi_n(\mathbf{D}_n)$, $n = 2m \geq 4$. As GL_n -module, we have $\varphi_n(\mathbf{D}_n) = \varphi_{n-1} \otimes \nu_{m-1} \oplus \cdots \oplus \varphi_1 \otimes \nu_{-m+1}$.

Theorem 8.3. *As GL_n -module, $\mathcal{O}(X) = \mathcal{S}(\varphi_n(\mathbf{D}_n))$.*

Proof. There are no invariants in this case, so we have to proceed a little differently. We first find a general point of X . Let e_1, \dots, e_n be the usual basis of \mathbb{C}^n . Let $\omega = e_2 \wedge e_3 + \cdots + e_{2n-2} \wedge e_{2n-1}$ considered as an element of the Lie algebra of \mathbf{D}_n . Then the action of $\exp(\omega)$ on e_1 sends it to the sum v of the elements $e_1 \wedge \omega^k \in \varphi_{2k+1}$, $k = 0, \dots, m-1$. The isotropy group H of SL_n acting

on v is the semidirect product of \mathbb{C}_{m-1} with $\text{Hom}(\mathbb{C} \cdot e_n, \mathbb{C}^{n-1}) \oplus \text{Hom}(\mathbb{C}^{n-2}, \mathbb{C} \cdot e_1)$ where \mathbb{C}^{n-2} here stands for the span of e_2, \dots, e_{n-1} and \mathbb{C}^{n-1} stands for $\mathbb{C}^{n-2} \oplus \mathbb{C} \cdot e_1$. Note that our copy of \mathbb{C}_{m-1} acts standardly on \mathbb{C}^{n-2} . Now $\dim \text{SL}_n/H = \dim X$, so that $\text{SL}_n \cdot v$ is a dense orbit in X . Since X is factorial [VP72, Theorem 4], any divisor in the complement of the dense orbit must be defined by a semi-invariant of SL_n , hence by an invariant. Thus there are no such divisors, so that the complement of $\text{SL}_n \cdot v$ has codimension 2. It follows that $\mathcal{O}(X) \simeq \mathcal{O}(\text{SL}_n/H)$. But the irreducibles of SL_n with an H -fixed vector are those of the form $\varphi_1^{a_1} \varphi_3^{a_3} \dots \varphi_{n-1}^{a_{n-1}}$ where the a_i are nonnegative, and the fixed point set has dimension one. Thus $\mathcal{O}(X)$ is as claimed. \square

We now compute the ring of K -invariants in the cases (6.3) and (6.4) of Table 2.

Proposition 8.4. *Let X (resp. Y) be the closure of the orbit of the highest weight vector of $\varphi_7(\text{D}_8)$ (resp. $\varphi_8(\text{D}_8)$). Consider the action of B_4 on X and Y where $\varphi_1(\text{D}_8)|_{\text{B}_4} = \varphi_4(\text{B}_4)$. Then $\mathcal{O}(X)^{\text{B}_4} = \mathbb{C}[f_4]$ and $\mathcal{O}(Y)^{\text{B}_4} = \mathbb{C}[f_2, f_3]$ where $\deg f_i = i$.*

Proof. Using LiE [vLCL92, vL94] one computes that the Poincaré series of $\mathcal{O}(X)^{\text{B}_4}$ is $1 + t^4 + \dots$ and that the Poincaré series of $\mathcal{O}(Y)^{\text{B}_4}$ is $1 + t^2 + t^3 + t^4 + t^5 + \dots$. Recall that X and Y are normal, hence so are $\mathcal{O}(X)^{\text{B}_4}$ and $\mathcal{O}(Y)^{\text{B}_4}$. Thus $\dim \mathcal{O}(Y)^{\text{B}_4} \geq 2$. The restriction of $\varphi_7(\text{D}_8)$ (resp. $\varphi_8(\text{D}_8)$) to B_4 is $\varphi_1 \varphi_4$ (resp. $\varphi_1^2 + \varphi_3$). Let P (resp. Q) be the stabilizer of the highest weight line in $\varphi_7(\text{D}_8)$ (resp. $\varphi_8(\text{D}_8)$). Then the Levi components $L(P)$ and $L(Q)$ of P and Q double cover representatives of the two SO_{16} -conjugacy classes of embeddings of GL_8 in SO_{16} . We have $L(P) \simeq (\text{SL}_8 \times \mathbb{C}^*)/(\mathbb{Z}/4\mathbb{Z})$ and the same for $L(Q)$. Restricted to $L(P)$, $\varphi_7(\text{D}_8)$ becomes the representation $\nu_4 + \wedge^6(\mathbb{C}^8) \otimes \nu_2 + \wedge^4(\mathbb{C}^8) + \wedge^2(\mathbb{C}^8) \otimes \nu_{-2} + \nu_{-4}$. The highest weight space of $\varphi_7(\text{D}_8)$ is ν_4 . The tangent space to X at a nonzero point of ν_4 is $\nu_4 + \wedge^6(\mathbb{C}^8) \otimes \nu_2$ so that $\dim X = 29$. The restriction of $\varphi_7(\text{D}_8)$ to $L(Q)$ is $\wedge^7(\mathbb{C}^8) \otimes \nu_3 + \wedge^5(\mathbb{C}^8) \otimes \nu_1 + \wedge^3(\mathbb{C}^8) \otimes \nu_{-1} + \mathbb{C}^8 \otimes \nu_{-3}$. For $\varphi_8(\text{D}_8)$, the decompositions relative to $L(P)$ and $L(Q)$ are reversed, so $\dim Y = 29$, also.

Consider the action of $H = \text{Ad SL}_3$ on \mathbb{C}^9 as $\varphi_1 \varphi_2 + \theta_1$. Then $\varphi_4(\text{B}_4)|_H = 2\varphi_1 \varphi_2$. Clearly the image of H in SO_{16} lies in a copy of GL_8 . Suppose that this copy of GL_8 is double covered by a conjugate of $L(P)$. Then $X^H \neq (0)$, and the B_4 -orbit of a nonzero fixed point is closed since the normalizer of H in B_4 is a finite extension of H [Lun75, 3.1 Corollary 1]. It is easy to check that the isotropy group of a nonzero point of X^H is at most a finite extension of H . Thus the dimension of the corresponding closed B_4 -orbit is 28. Hence $\dim \mathcal{O}(X)^{\text{B}_4} \leq 1$ and the Poincaré series information gives that $\mathcal{O}(X)^{\text{B}_4} = \mathbb{C}[f_4]$ where $\deg f_4 = 4$. If our copy of GL_8 were double covered by a conjugate of $L(Q)$, then we would see that $\dim \mathcal{O}(Y)^{\text{B}_4} \leq 1$, which is a contradiction. Thus $\mathcal{O}(X)^{\text{B}_4}$ is as claimed.

Now consider the group $K = \text{SO}_6 \times \text{SO}_3 \subset \text{SO}_9$. Then the double cover \tilde{K} of K is $(\text{SL}_4 \times \text{SL}_2)/\pm I$ and $\varphi_4(\text{B}_4)$, as \tilde{K} -representation, is $\mathbb{C}^4 \otimes \mathbb{C}^2 + (\mathbb{C}^4)^* \otimes \mathbb{C}^2$. Thus \tilde{K} is a subgroup of a copy of GL_8 in SO_{16} . If this GL_8 is double covered by a conjugate of $L(P)$, then one sees that there are no nonzero fixed points of \tilde{K} (actually K) in $\varphi_8(\text{D}_8)$. But $\varphi_8(\text{D}_8)|_{\text{B}_4} = \varphi_1^2 + \varphi_3$ has K -fixed points of dimension 2. Hence our copy of GL_8 is double covered by a conjugate of $L(Q)$ and the weight space ν_4 of the restriction of $\varphi_8(\text{D}_8)$ to $L(Q)$ lies in Y and is fixed by \tilde{K} . The group $N_{\text{B}_4}(\tilde{K})/\tilde{K} \simeq \mathbb{Z}/2$ flips the highest and lowest weight spaces $\nu_{\pm 4}$. Since \tilde{K} is a maximal connected reductive subgroup of B_4 , the stabilizer of ν_4 is \tilde{K} and any point of ν_4 lies on a closed orbit. The slice representation of \tilde{K} is $S^2(\mathbb{C}^4) + \theta_1$ which shows that the principal isotropy group H of the action of B_4 (actually SO_9) is $\text{SO}_3 \times \text{SO}_3 \times \text{SO}_3$. It follows that $\dim Y/\text{B}_4 = 2$. Now $N_{\text{SO}_9}(H)/H \simeq W(\text{D}_3)$, the Weyl group of D_3 , where $V := \varphi_8(\text{D}_8)^H$ has dimension 5. One easily computes that the generators of $\mathcal{O}(V)^{W(\text{D}_3)}$ are of degree at most 5. Then by the Luna-Richardson theorem [Lun75, 3.2 Corollary] it follows that the invariants of $\mathcal{O}(Y)^{\text{SO}_9}$ have generators in degree at most 5, and then from our information about the Poincaré series it follows that $\mathcal{O}(Y)^{\text{SO}_9} = \mathbb{C}[f_2, f_3]$ where $\deg f_i = i$, $i = 2, 3$. \square

Remark 8.5. There is no representation of SO_9 with principal isotropy group $H = \mathrm{SO}_3 \times \mathrm{SO}_3 \times \mathrm{SO}_3$ and slice representation $S^2(\mathbb{C}^4) + \theta_1$ of $K = \mathrm{SO}_6 \times \mathrm{SO}_3$ which has homogeneous invariants f_2 and f_3 of degrees 2 and 3, respectively. The reason is that we would have a slice which is an open K -invariant subset of the linear subspace $V = \mathbb{C} \cdot v + S^2(\mathbb{C}^4)$ where K fixes v , and the restrictions of the f_i to V would have to be functions of $\mathbb{C} \cdot v$ alone since the invariant of $S^2(\mathbb{C}^4)$ is of degree 4. Thus f_2 and f_3 would be algebraically dependent, a contradiction to normality.

Remark 8.6. The generators f_2 and f_3 form a homogeneous regular sequence in $\mathcal{O}(Y)$, hence $\mathcal{O}(Y)$ is a free graded $\mathbb{C}[f_2, f_3]$ -module [Sta79, Lemma 3.3]. It follows that $\mathcal{O}(Y)$ is cofree, i.e., each module of covariants is free over $\mathbb{C}[f_2, f_3]$. Of course, we have the analogous result for $\mathcal{O}(X)$.

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